# Signals, Systems, & Networks

SHANMUGA

Based on Principles of
Linear Systems and Signals
by BP Lathi

# ES 216 Signals, Systems, Networks

## Signals

- Function of time|space
- Domain (Ind. var.) eg. time|space
- Range (Dep. var.) eg. volt|current

Extent of signal. 1D

Indep. var. Origin dest.

$$\left.\begin{array}{l} - \quad -\infty < \text{time}\,(t) < \infty \\ - \quad -\infty < \text{space}\,(x) < \infty \end{array}\right\} \text{Real Line}$$

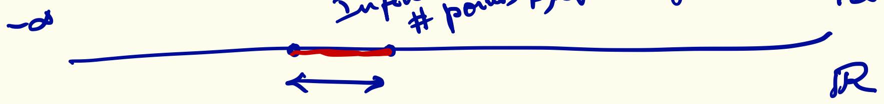
Countable

Real $\mathbb{R}$ = Rational $\cup$ Irrational

$p/q$ $q \neq 0$

Uncountable
Infinite
\# points

$p, q$ integers



Countable

$\Rightarrow$ one-one mapping with integers

Cantor's theorem

1. a) Continuous-time Signals (Physical)

- $\overset{\text{time}}{t}$ → entire or part of real line. (Domain)

Sampling

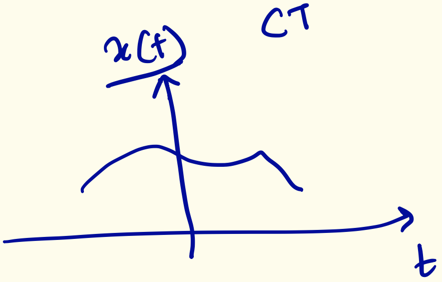
eg. voltage in a resistor.

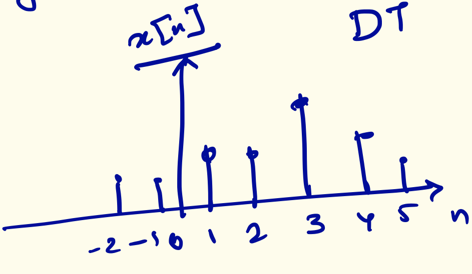
b) Discrete-Time Signals (Computer / Digital)

- $\overset{\text{time}}{n}$ → entire or part of integer line. (Domain) Countable set

eg. Stock market daily average.

rainfall per day over an year.





2. Tocci DAC|ADC

a) Analog $x(t)$

— range takes on continuous values. (part of real line)

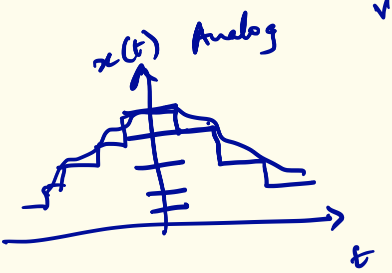
Def. var. quantization

eg. A/C current

b) Digital

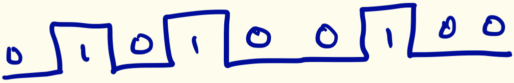
$x(t)$

— range takes on discrete values. (part of integer/ rational line)



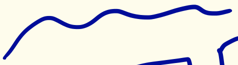
eg. Binary digital stream

M = 2

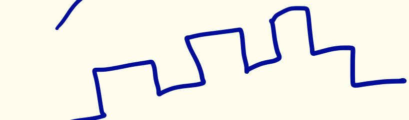


Signal

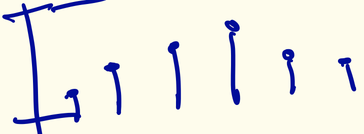
— ✓CT, A



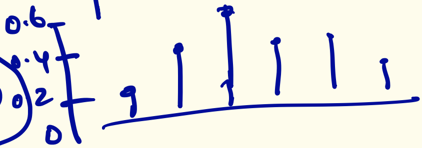
— CT, D



— ✓DT, A



Computer — DT, D



3. a. Energy

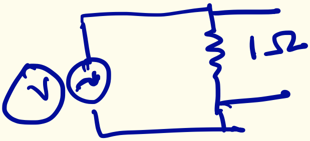
x(t)

$$E_x = \int_{-\infty}^{+\infty} x^2(t)\, dt$$

$x(t) \to 0$ as $t \to \infty$, $P_x = 0$

$x(t) \to V$

Unit → $E_x$ → $V^2 s$



Energy in $T_s$

$T_s$ seconds

$$E_x = \int_0 x^2(t)\, dt$$

b. Power

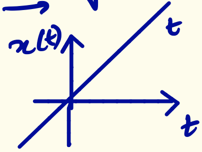
$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_T x^2(t)\, dt$$

eg. Periodic, statistical regularity → random

$x(t) = \sin t \quad -\infty < t < \infty$

$E_x = \infty$ Unit of $P_x \to V^2$

No $E_x$, No $P_x$ $x(t) = t \quad -\infty < t < \infty$



4. a) Deterministic

eg. $x(t) = \cos t$

b) Random Process (Infinite #) ensemble of

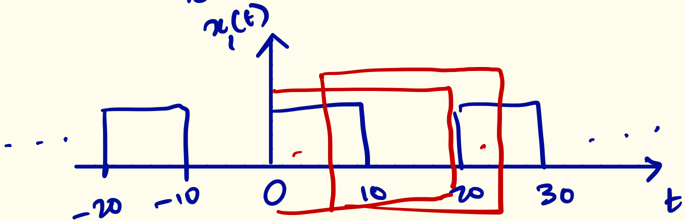
eg. $\underline{X(t)} \to$ signals

5. a) Periodic

$$x(t) = x(t+T_0)$$

$T_0 \rightarrow$ Period

eg.



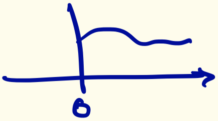
$T_0 = 20$

eg. $x_2(t) = \sin 2t$ $\quad T_0 = \pi$

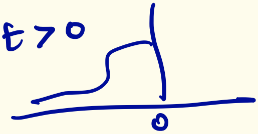
$x_3(t) = \cos 5\pi t$ $\quad T_0 = \frac{2}{5}$

b) Aperiodic

6. a) Causal $\quad x(t) = 0,\ t<0$

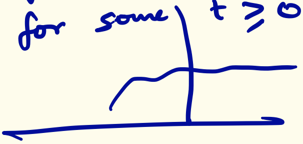


b) Anti-Causal $\quad x(t) = 0,\ t>0$



c) Non-Causal $\quad x(t) \neq 0$ for some $t<0$

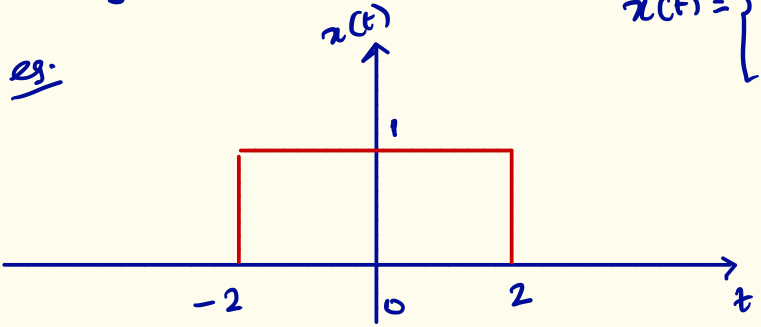
for some $t \geq 0$



# Signal Operations

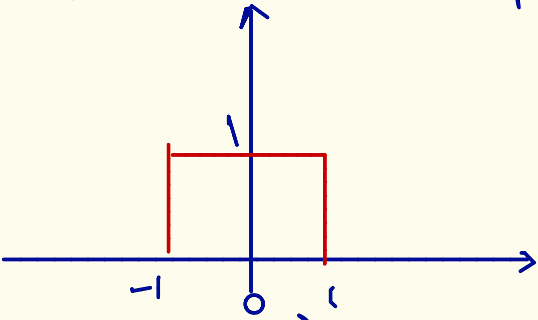
## 1. Scaling

eg.



$$x(t) = \begin{cases} 1 & -2<t<2 \\ 0 & o/w \end{cases}$$

$y_1(t) = x(2t)$

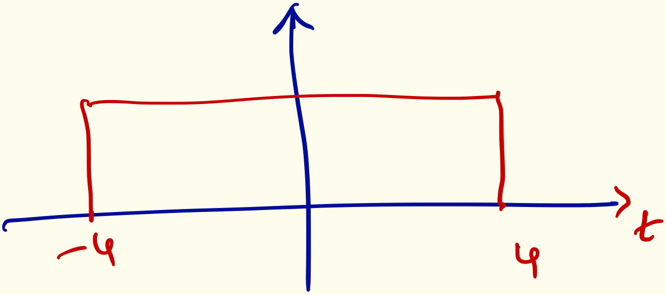


$$y_1(t) = x(2t) = \begin{cases} 1, & -2<2t<2 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} 1, & -1<t<1 \\ 0, & o/w \end{cases}$$

$y_2(t) = x(t/2)$

$$y_3(t) = \begin{cases} 1, & -4<t<4 \\ 0, & o/w \end{cases}$$

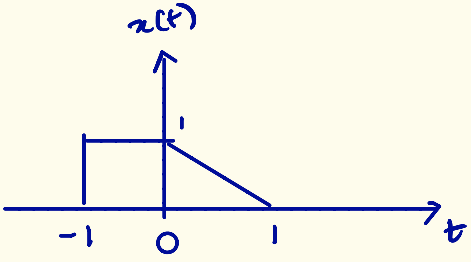


$x(at) \longrightarrow$ Compression if $|a|>1$

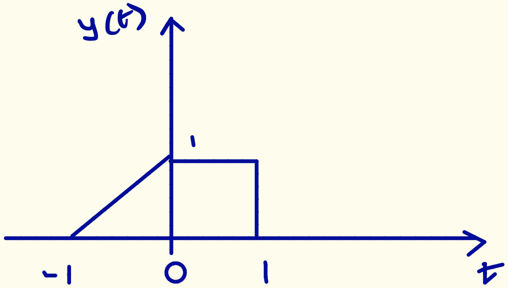
$x(at) \longrightarrow$ Expansion if $|a|<1$

2. Inversion

$x(t)$



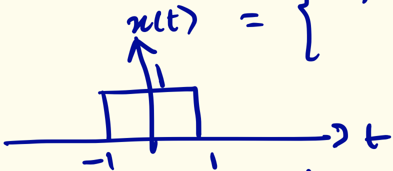
$y(t) = x(-t)$



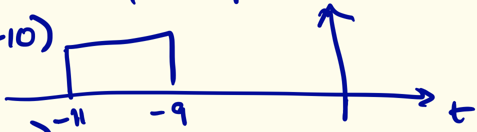
$x(t) = \begin{cases} 1, & -1<t<1 \end{cases}$

3. Shifting

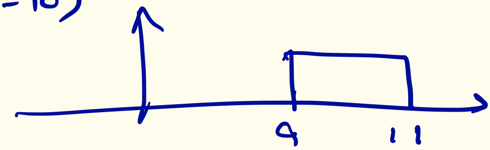
$x(t)$



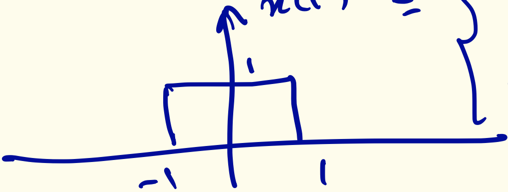
Advance $y_1(t) = x(t+10)$



Delay $y_2(t) = x(t-10)$

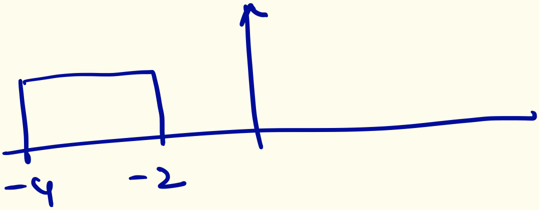


$$y(t) = x(at+b) = x\left(a\left(t+\frac{b}{a}\right)\right)$$



$$x(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & o/w \end{cases}$$

$$y(t) = x(2t+3) = \begin{cases} 1, & -1 < 2t+3 < 1 \\ 0, & o/w \end{cases}$$



$$= \begin{cases} 1, & -2 < t < -1 \\ 0, & o/w \end{cases}$$

# Three Fundamental Signals CT

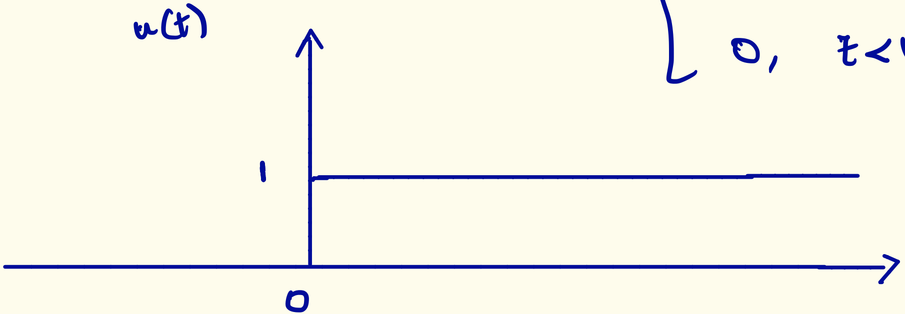
1. Unit Step  $u(t)$

2. Dirac Delta | Unit Impulse  $\delta(t)$

3. Complex Exponential  $e^{st}$, $s=\sigma+j\omega$

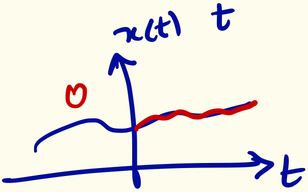
1. Unit Step

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Causal

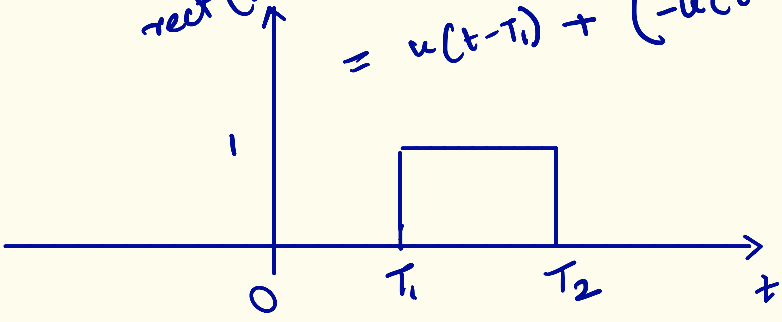
$x(t) \times u(t) \rightarrow$ Causal

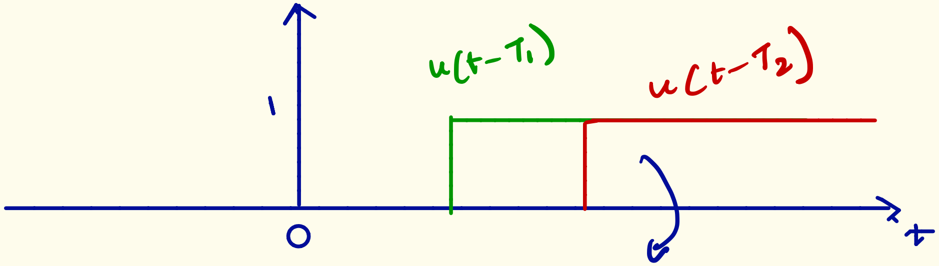


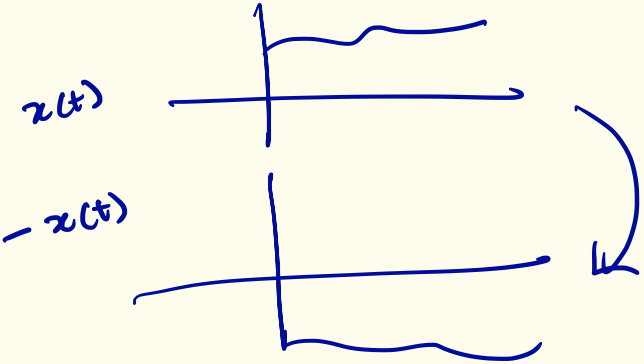
Rectangular

$$\text{rect}(t) = u(t-T_1) \ominus u(t-T_2)$$

$$= u(t-T_1) + (-u(t-T_2))$$





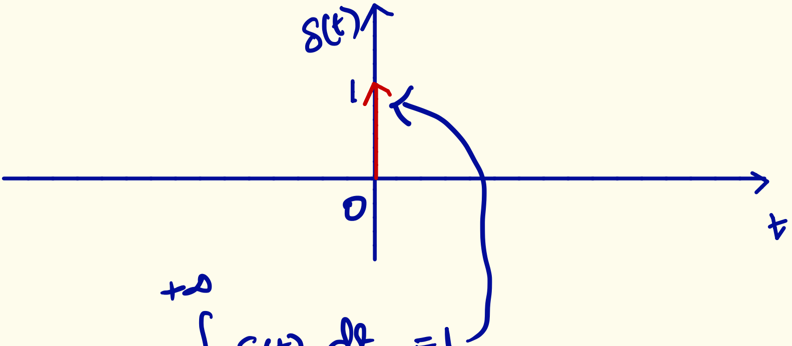


Amplitude scaling
$-1$

## 2. Dirac Delta | Unit Impulse

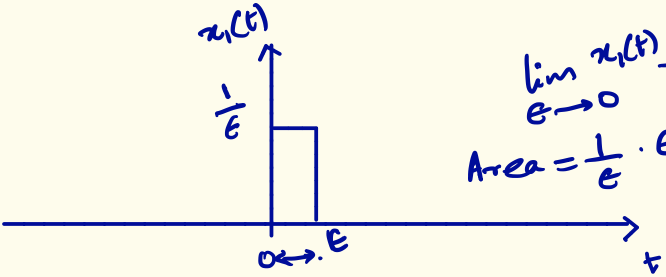
– Generalized function

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$



Area $\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$

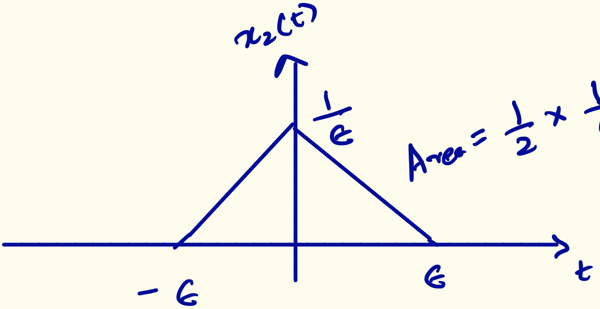
a)



$\lim_{\epsilon \to 0} x_1(t) \longrightarrow \delta(t)$

Area $= \frac{1}{\epsilon} \cdot \epsilon = 1$

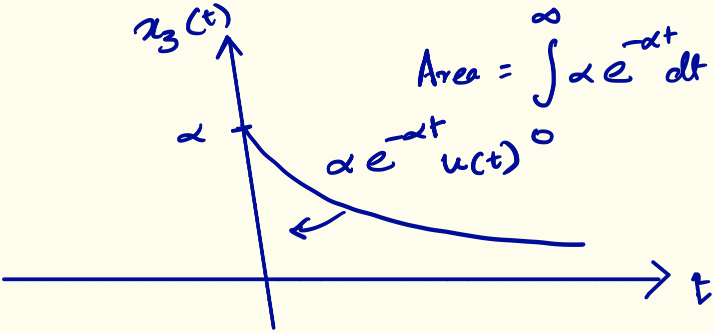
b)



$$\text{Area} = \frac{1}{2} \times \frac{1}{\epsilon} \cdot 2\epsilon = 1$$

$$\lim_{\epsilon \to 0} x_2(t) = \delta(t)$$

c)



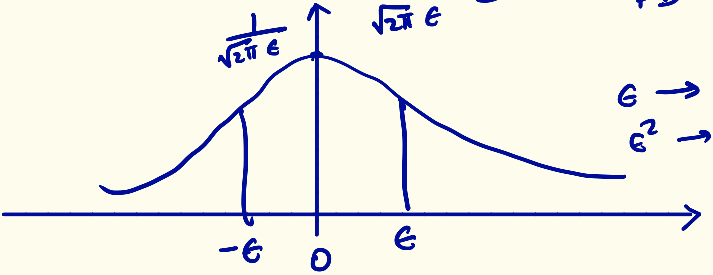
$$\text{Area} = \int_0^\infty \alpha e^{-\alpha t} dt = 1$$

$$\lim_{\alpha \to \infty} x_3(t) = \delta(t)$$

d)

$$x_4(t) = \frac{1}{\sqrt{2\pi}\,\epsilon} e^{-\frac{t^2}{2\epsilon^2}}$$

PDF $\Rightarrow$ Area $= 1$



$\epsilon \rightarrow$ std. dev.

$\epsilon^2 \rightarrow$ variance

$$\lim_{\epsilon \to 0} x_4(t) = \delta(t)$$

# Relation between $u(t)$ and $\delta(t)$

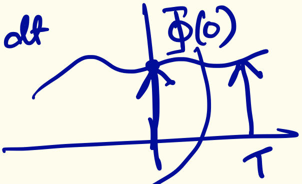
Signal $\Phi(t)$

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undef}, & t = 0 \end{cases}$$

$$\hat{\Phi}(t) = \Phi(t)\,\delta(t) = \Phi(0)\,\delta(t)$$

$$\int_{-\infty}^{+\infty} \hat{\Phi}(t)\,dt = \int_{-\infty}^{+\infty} \Phi(0)\,\delta(t)\,dt$$

$$= \Phi(0) \int_{-\infty}^{+\infty} \delta(t)\,dt$$



$$= \Phi(0)$$

$$\int_{-\infty}^{+\infty} \Phi(t)\,\delta(t-T)\,dt = \int_{-\infty}^{+\infty} \Phi(T)\,\delta(t-T)\,dt$$

$$= \Phi(T)$$

→ Sampling Property

$$\delta(t-T) = \begin{cases} \text{undef}, & t = T \\ 0, & t \neq T \end{cases}$$

$$\int_{-\infty}^{+\infty} \underset{dv}{\overset{\text{Unit step}}{\frac{du(t)}{dt}}} \underset{u}{\overset{\text{any signal}}{\Phi(t)}}\, dt \qquad u(\infty)=1 \quad u(-\infty)=0$$

$$= u(t)\,\Phi(t)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u(t)\,\dot{\Phi}(t)$$

$$= \overset{1}{u(\infty)}\,\Phi(\infty) - \overset{0}{u(-\infty)}\,\Phi(-\infty) - \int_{0}^{\infty} \dot{\Phi}(t)\,u(t)$$

$$= \Phi(\infty) - \int_{0}^{\infty} \dot{\Phi}(t)$$

$$= \Phi(\infty) - \Phi(t)\Big|_{0}^{\infty}$$

$$= \Phi(\infty) - \left[\Phi(\infty) - \Phi(0)\right]$$

$$= \Phi(0) \longrightarrow ②$$

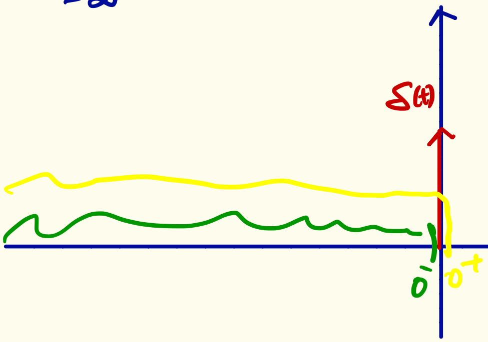
$$\int_{-\infty}^{+\infty} \delta(t)\,\Phi(t)\,dt = \Phi(0) \longrightarrow ①$$

from ① & ②

$$\frac{du(t)}{dt} = \delta(t)$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$



$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$$

$$u(0^-) = \int_{-\infty}^{0^-} \delta(\tau)\, d\tau = 0$$

$$u(0^+) = \int_{-\infty}^{0^+} \delta(\tau)\, d\tau = 1$$

## 3. Complex Exponential

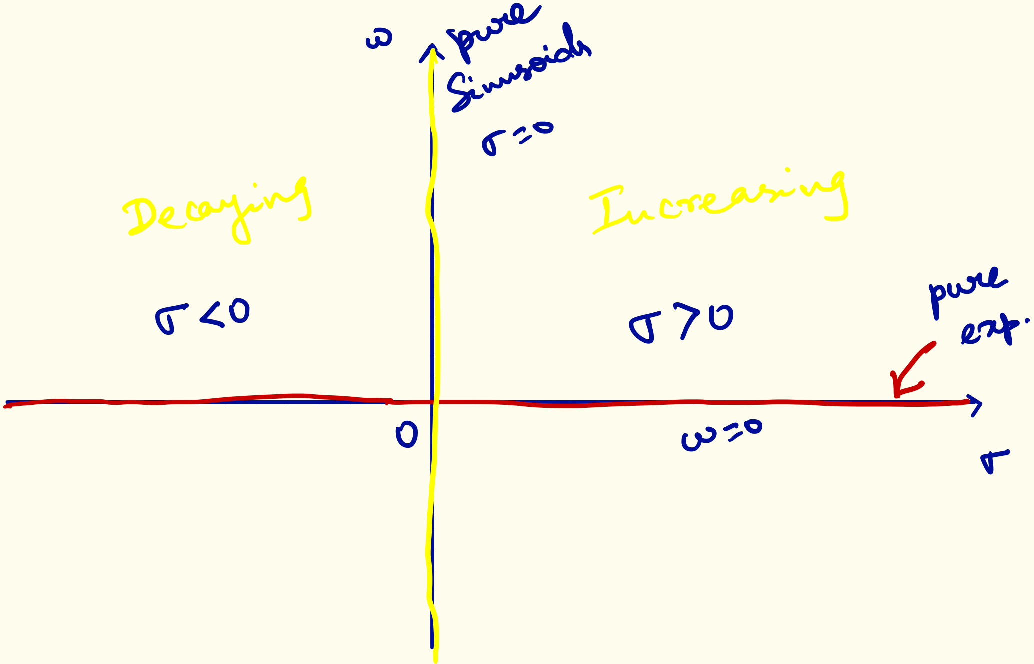
$$e^{st} \quad , \quad s = \sigma + j\omega$$

a) $\omega = 0$ , $e^{\sigma t}$

b) $\sigma = 0$ , $e^{j\omega t} = \cos\omega t + j\sin\omega t$

c) $\sigma \neq 0$ , $\omega \neq 0$ $\quad e^{\sigma t} e^{j\omega t}$

$$= e^{\sigma t}\left[\cos\omega t + j\sin\omega t\right]$$



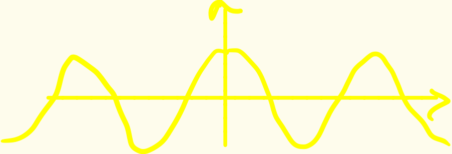
$$\frac{1}{2}\left[e^{st} + e^{\overset{\leftarrow \text{Complex conjugate}}{s^*}t}\right] = \frac{1}{2}e^{\sigma t}\left[e^{j\omega t} + e^{-j\omega t}\right]$$

$$= \frac{1}{2}e^{\sigma t}\left[\cos\omega t + j\cancel{\sin}\,\omega t + \cos\omega t - j\cancel{\sin}\,\omega t\right]$$

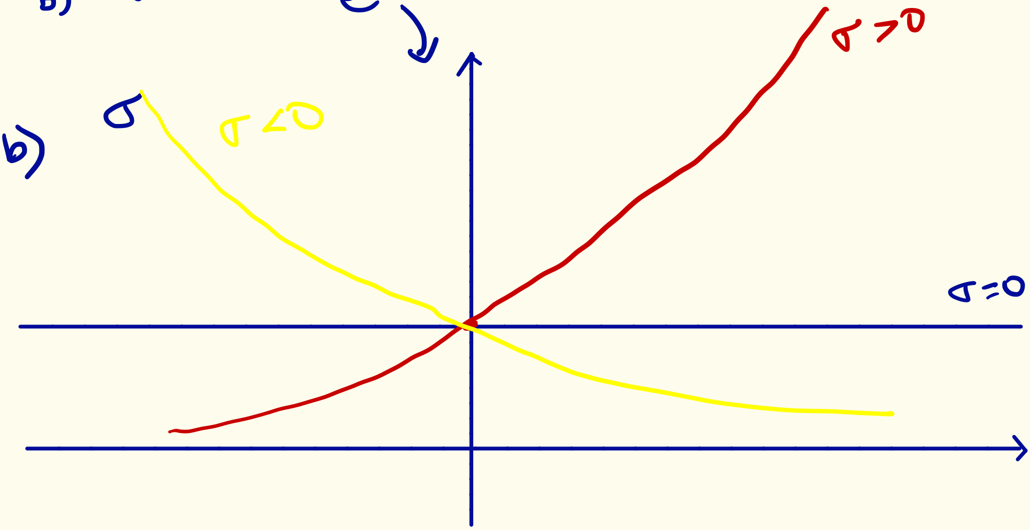
$$= \frac{1}{2}\,e^{\sigma t}\left[2\cos\omega t\right]$$

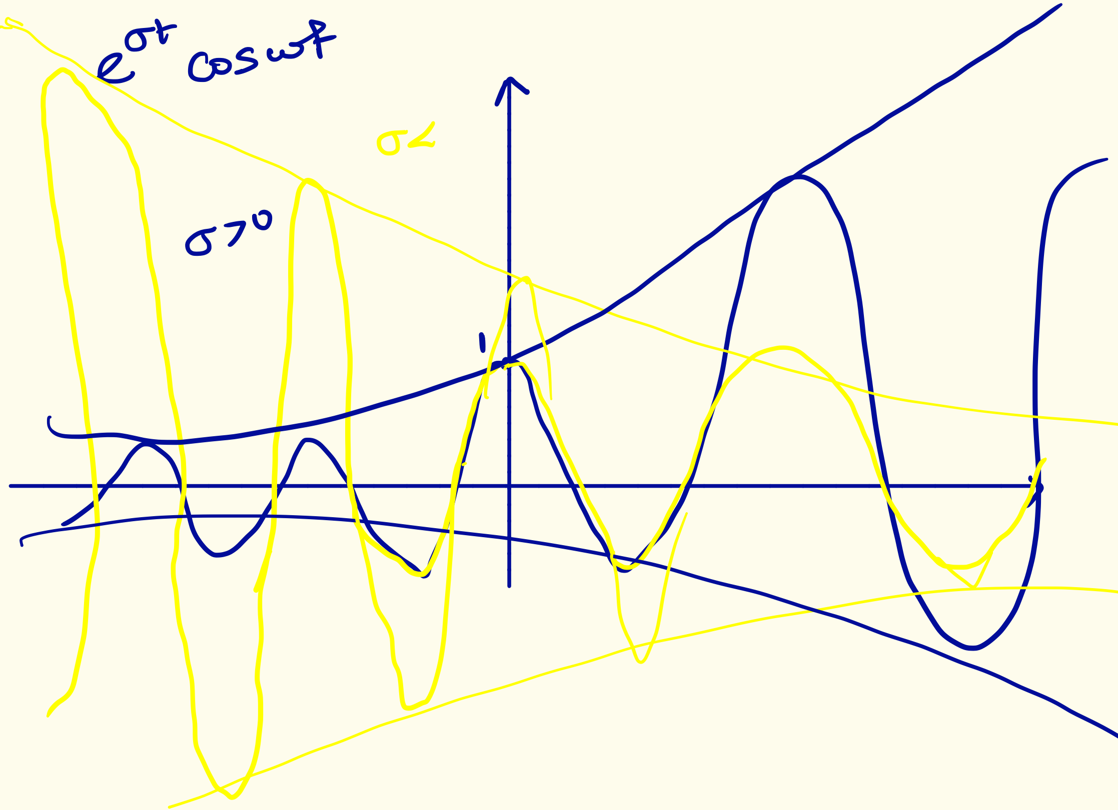
$$= e^{\sigma t}\cos\omega t$$

a): $\sigma = 0$ $\cos\omega t$



b) $\omega = 0$ $e^{\sigma t}$




$e^{\sigma t} \cos \omega t$
$\sigma<$
$\sigma>0$
1

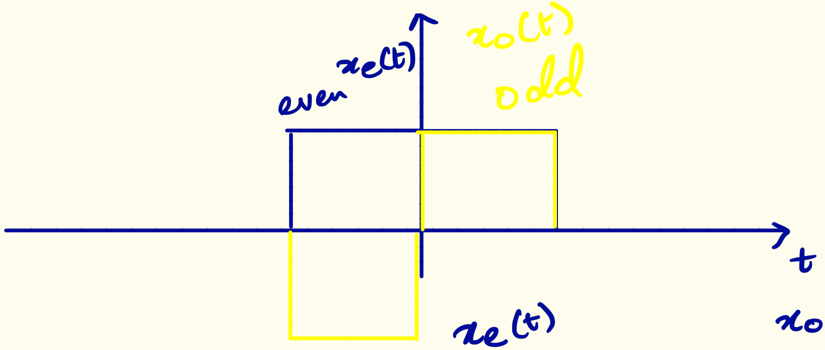
# Even | odd Signals

Even $f(t) = f(-t)$ Symmetric

eg. $\cos(\omega t)$, $|t|$, $t^2$, Gaussian

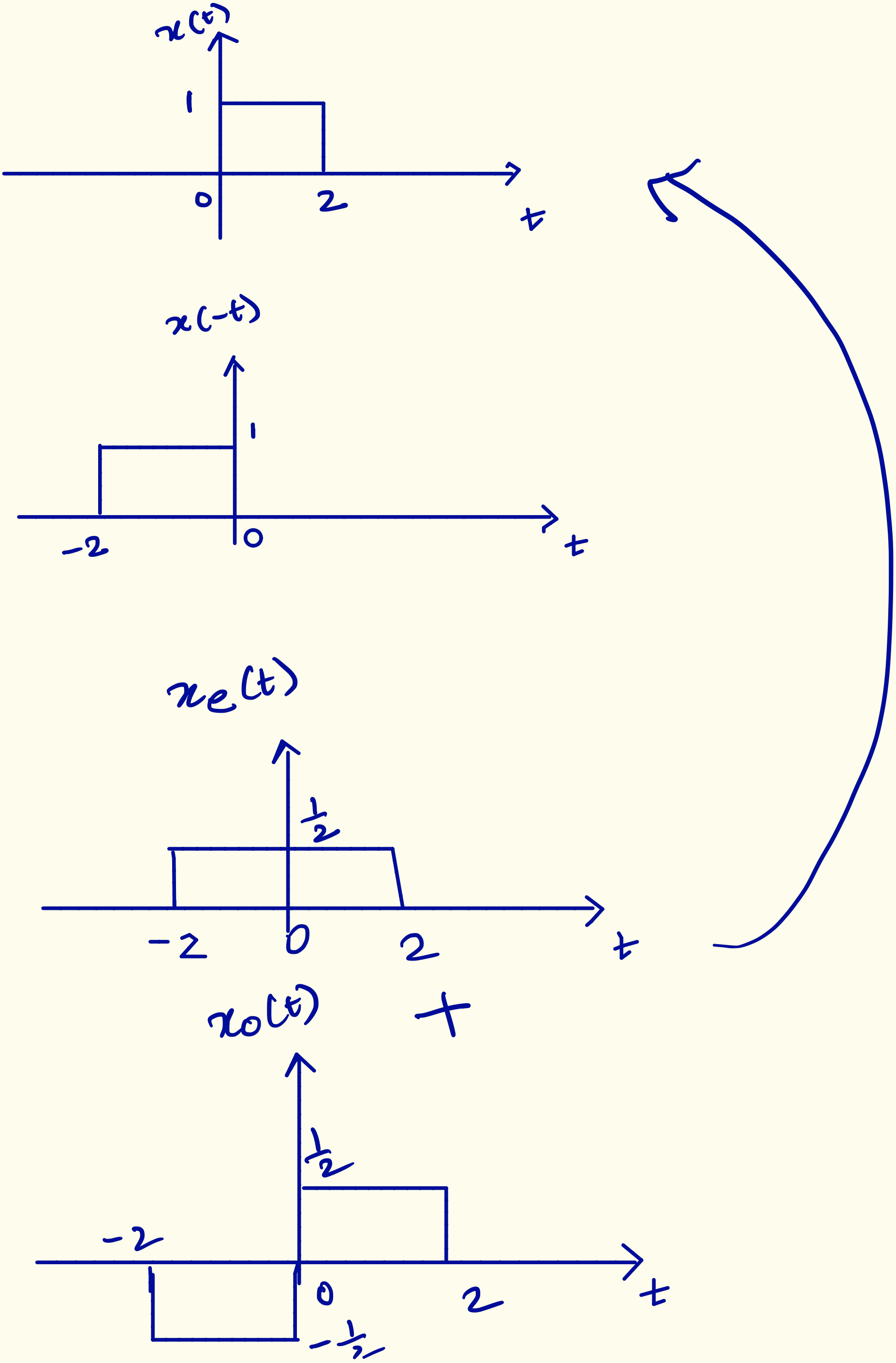
Odd $f(t) = -f(-t)$ Anti symmetric

eg. $\sin \omega t$, $t$, $t^3$



$$x(t) = \overbrace{\frac{1}{2}\left[x(t) + x(-t)\right]}^{x_e(t)} + \overbrace{\frac{1}{2}\left[x(t) - x(-t)\right]}^{x_o(t)}$$

$x_e(-t) = x_e(t)$ $x_o(-t) = -x_o(t)$


x(t)
1
0
2
t
x(-t)
1
-2
0
t
$x_e(t)$
$\frac{1}{2}$
-2
0
2
t
+
$x_o(t)$
$\frac{1}{2}$
-2
0
2
t
$-\frac{1}{2}$

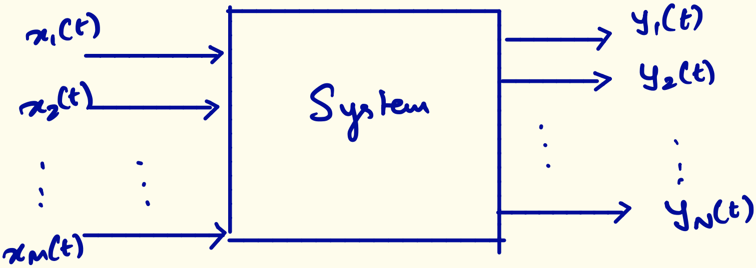
even x even = even

odd x odd = even

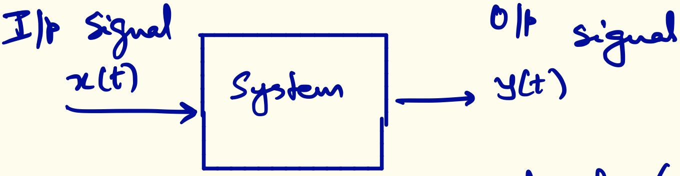
odd x even = odd

# Systems

- Processes Signals



Multiple Input Multiple Output (MIMO)

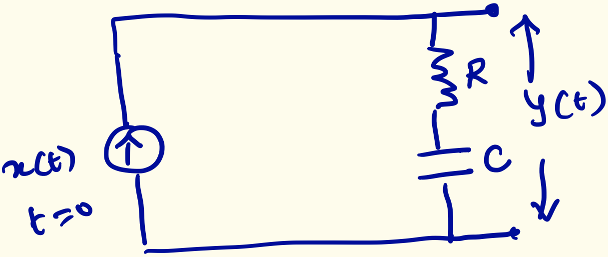


Single Input Single Output (SISO)

# Classification of Systems

## a) Linear and Non-linear

eg. R-C circuit



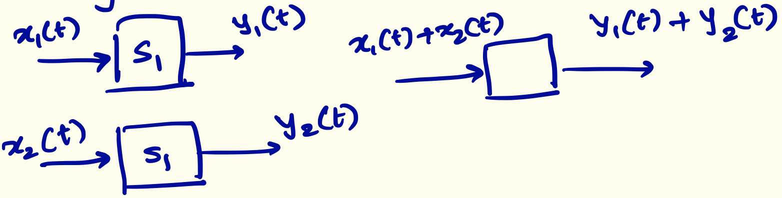
$$y(t) = R\,x(t) + \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

$$= R\,x(t) + \underbrace{\frac{1}{C}\int_{-\infty}^{0} \tilde{x}(\tau)\,d\tau}_{v_c(0^-)} + \frac{1}{C}\int_{0}^{t} x(\tau)\,d\tau$$

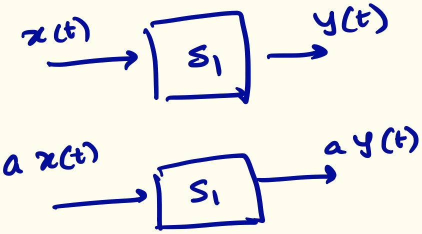
$$y(t) = \underbrace{v_c(0^-)}_{\substack{\text{zero-input}\\ \text{response}\\ y_{ZI}(t)}} + \underbrace{R\,x(t) + \frac{1}{C}\int_{0}^{t} x(\tau)\,d\tau}_{\substack{\text{zero-state}\\ \text{response}\\ y_{ZS}(t)}}$$

# Linear System

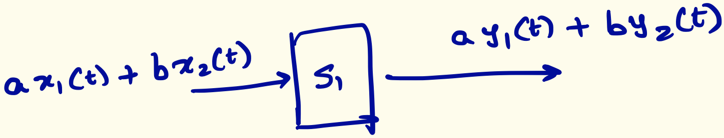
a) Additivity



b) Homogeneity | scaling



Superposition = Additivity + Homogeneity



$x(t)$ and $y(t)$ are governed by a linear ODE with constant / function of time coefficients.

Non-linear

eg. $y(t) = \log(x(t))$

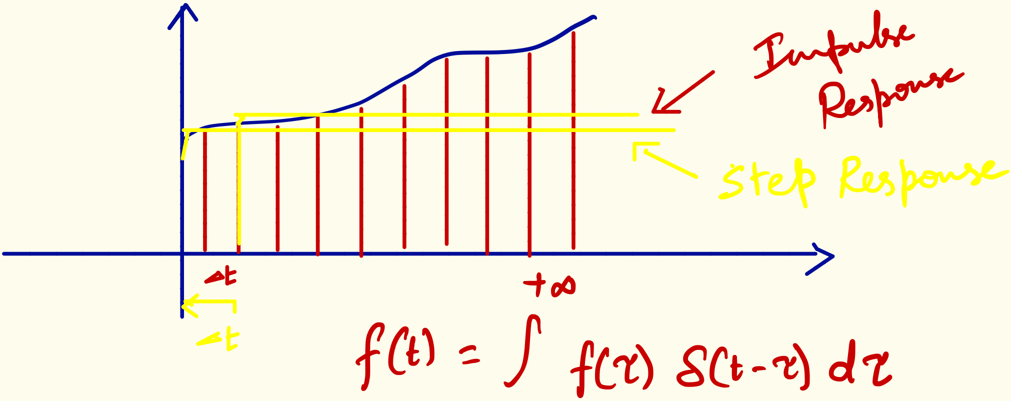
$y(t) = x^2(t)$

$y(t) = e^{x(t)}$

$y(t) = t\,x(t) \longrightarrow$ linear

$y(t) = (t+2)\,x(t) \longrightarrow$ linear

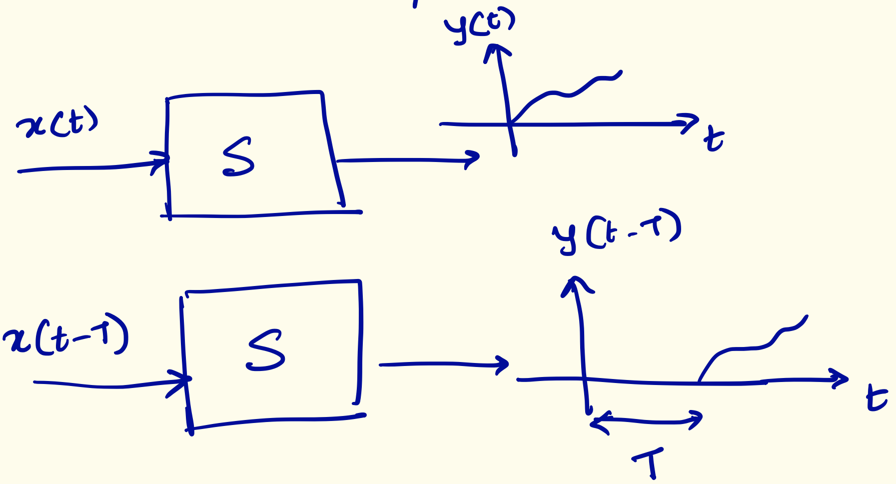
$y(t) = x(t) + 5 \longrightarrow$ Non-linear

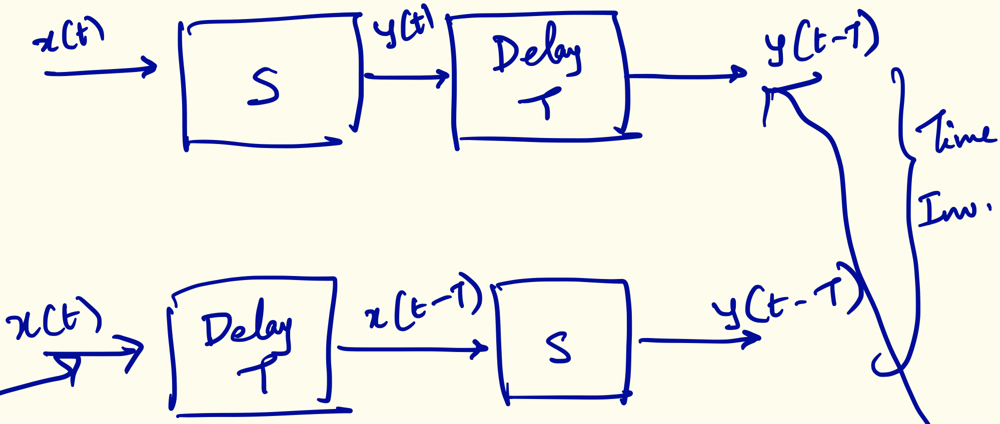


$$f(t) = \int_{-\infty}^{} f(\tau)\,\delta(t-\tau)\,d\tau$$

$-\infty < t < \infty$

## 2. Time Variant / Invariant





eg.

$y(t) = x(t-10)$ TI

$y(t) = t\,x(t)$ TV

$y(t-T) = (t-T)\,x(t-T)$

$t\,x(t-T)$

# Linear Time Invariant (LTI)

_ Linear ODE with constant coeff.

eg.

$$3\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = 3x(t)$$

Non-linear, time invariant

eg. $y(t) = x^2(t)$

$y(t) = e^{x(t)}$

3. Continuous time | Discrete Time

CT signals

A/D

DT signals

A/D

In this course

CTA

DTA

4. Analog | Digital

CT A

eg. Film Camera

DT D

eg. Computer

## 5. Instantaneous | Dynamic

| | |
|---|---|
| Present O/P depends only on present I/P | Present O/P depends on present, past, future I/Ps |
| Memory-less | Systems with Memory |
| eg: $y(t) = x^2(t)$ | eg. $y(t) = x^2(t+4) + x^3(t) - 3x^4(t-2)$ |

↓ special case

Pr, past

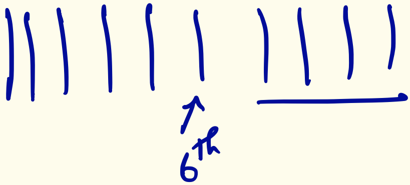
pr, past, future

## 6. Causal | Non-Causal

| | |
|---|---|
| Present O/P depends on past and present I/Ps. | Present o/p depends on future I/ps in addition. |
| eg. $y(t) = x(t-3) + 3x(t)$ | eg. $y(t) = x(t) + \underline{x(t+5)}$ |
| 10 7 10 | future |
| Physically Realizable | Not realizable |

# Why Non-Causal ?

1. Pre-recorded time signal processing.

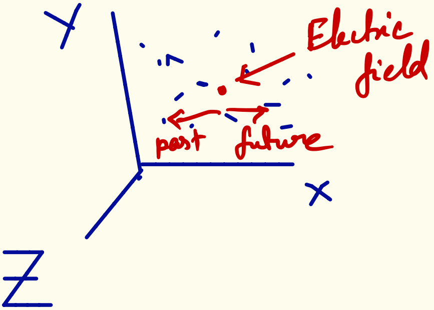


2. When independent variable is space

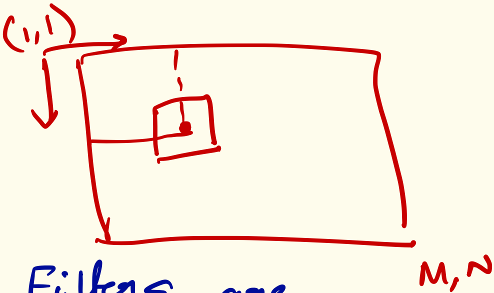
$f(x,y,z)$

$$\nabla . E = -\frac{\rho}{\epsilon_0}$$

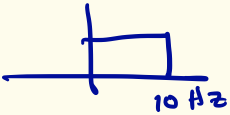
$$\nabla^2 \Phi = \frac{\rho}{\epsilon_0}$$



$$E = -\nabla \Phi$$



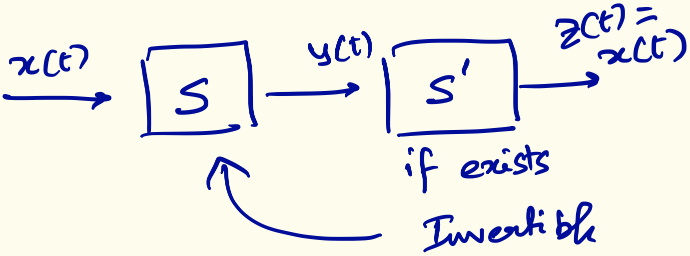
3. Ideal Filters are non-causal



7. Stable | Unstable

| Stable | Unstable |
|---|---|
| Bounded I/P → Bounded O/P | Bounded I/P → Un bounded O/P |

8. Invertible | Non-invertible



$$S' = S^{-1}$$

<u>eg:</u> Inv.

a) $y(t) = x(t+4)$

$z(t) = y(t-4)$

$z(t) = x(t)$

b) $y(t) = \frac{1}{a} x(t)$

$z(t) = a\, y(t)$

$z(t) = x(t)$

c) $y(t) = e^{x(t)}$

$z(t) = \ln(y(t)$

$z(t) = x(t)$

**Non-inv.**

$y(t) = x^2(t)$

$y(t) = |x(t)|$

# Time Domain Analysis LTIC

$$\left(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N\right) y(t)$$

$$= \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N\right) x(t)$$

$D = \frac{d}{dt}$ $\quad M \le N$

$$Q(D)\, y(t) = P(D)\, x(t) \Rightarrow y(t) = \frac{P(D)}{Q(D)} x(t)$$

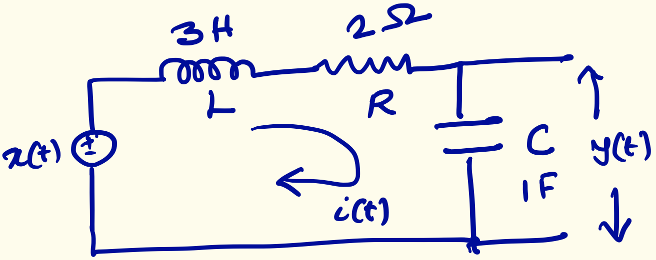
Linear ODE with Const. Coeff.

$y(t) \rightarrow$ Output

$x(t) \rightarrow$ input

$a_i, b_i \rightarrow$ Real constants

eg.



$$x(t) - 3\frac{di}{dt} - \frac{1}{1}\int i\,dt - 2i = 0$$

$$\frac{dx(t)}{dt} = 3\frac{d^2 i(t)}{dt^2} + 2\frac{di(t)}{dt} + i(t)$$

$$\int_{-\infty}^{t} i\,dt = y(t)$$

$$i(t) = \frac{dy}{dt}$$

$$\frac{dx(t)}{dt} = 3\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt}$$

$$Y(s) = f(s)\;X(s) \leftarrow \text{HPF}$$

$$Y(s) = f(s^{-1})\;X(s) \longleftarrow \text{LPF}$$

Diff. → HPF

Integ. → LPF

## Solution

Total Response = Zero Input Response + Zero State Response

$$y(t) = y_0(t) + y_{zs}(t)$$

$$-\infty < t < \infty$$

# Zero Input Response $y_0(t)$

$$x(t) = 0$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_0(t) = 0$$

$$Q(D)\, y_0(t) = 0$$

Homogeneous ODE

$$y_0(t) = c e^{\lambda t}$$

$$D^N(c e^{\lambda t}) = \underline{\lambda^N}\, c e^{\lambda t}$$

$$\vdots$$

$$D(c e^{\lambda t}) = \underline{\lambda}\, c e^{\lambda t}$$

$$(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N)\, y_0(t) = 0$$

$$Q(\lambda)\, y_0(t) = 0$$

$Q(\lambda) \longrightarrow N$ roots

## Case 1

$Q(\lambda)$ has distinct roots.

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_N e^{\lambda_N t}$$

## Case 2

$Q(\lambda)$ has one root with 'r' repetitions.

$$y_0(t) = (c_1 + c_2 t + \dots + c_r t^{r-1}) e^{\lambda_r t} + c_{r+1} e^{\lambda_{r+1} t} + \dots + c_N e^{\lambda_N t}$$

## Case 3

$Q(\lambda)$ has one pair of complex roots.

Complex conjugate roots $\alpha + j\beta$, $\alpha - j\beta$

$$y_0(t) = \underbrace{c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}} + \dots + c_N e^{\lambda_N t}$$

$$e^{\alpha t}\left[c_1 e^{j\beta t} + c_2 e^{-j\beta t}\right]$$

$$= \frac{c}{2} e^{\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right]$$

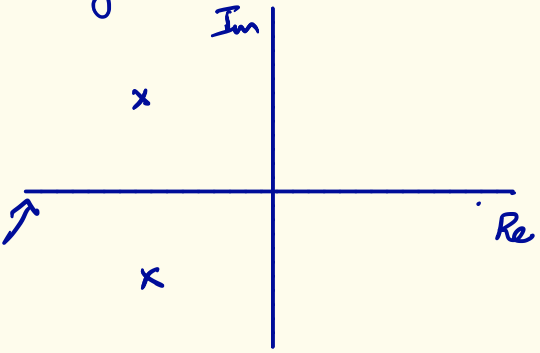
$$= \overset{\checkmark}{c} e^{\alpha t}\left[\cos(\beta t + \overset{\checkmark}{\theta})\right]$$

$$c_1 = \frac{c}{2} e^{j\theta}$$

$$c_2 = \frac{c}{2} e^{-j\theta}$$

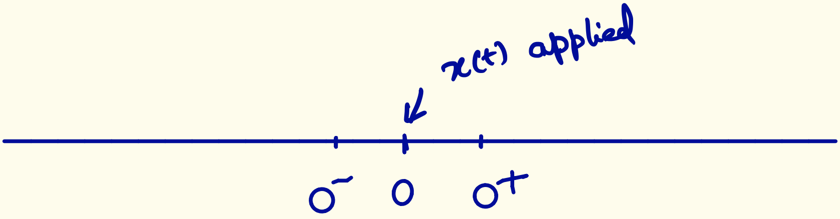
$\lambda_1, \lambda_2, \ldots, \lambda_N \rightarrow$ characteristic values

eigenvalues



$e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_N t}$

$\rightarrow$ Characteristic Modes



$y_0(t)$ at $t = 0^-, 0^+$

$$y_0(0^-) = y_0(0^+)$$

$$y(0^-) \neq y(0^+)$$

$$= y_0(0^-) \qquad = y_0(0^+) + y_{zs}(0^+)$$

eg.

$$(D^2+3D+2)\,y(t) = 4x(t)$$

$$\underline{y_0(0) = 2,\ \dot{y}_0(0) = -5}$$

$$y_0(t) = c_1 e^{-2t} + c_2 e^{-t}$$

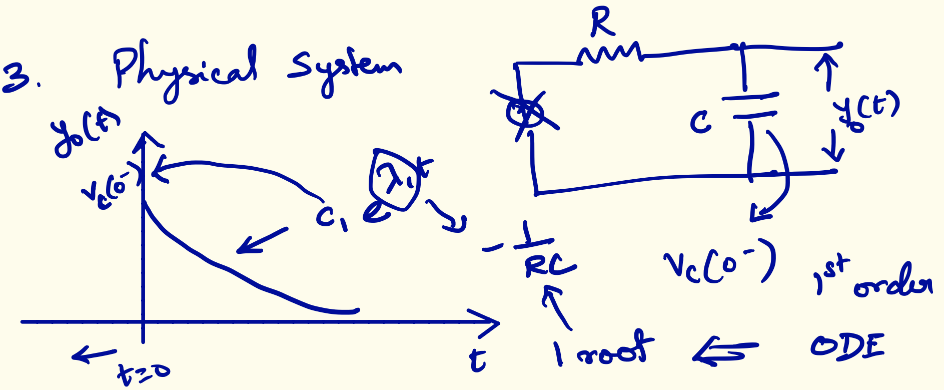
$$y_0(t) = 3e^{-2t} - e^{-t}$$

## Zero Input Response

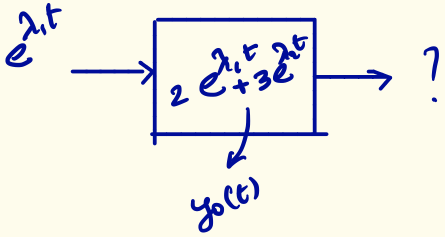
1. To estimate the coefficients of modes, we need auxiliary conditions.
   If given at $t=0$ ⇒ Initial conditions

2. $y_0(0^-) = y_0(0^+)$

3. Physical System



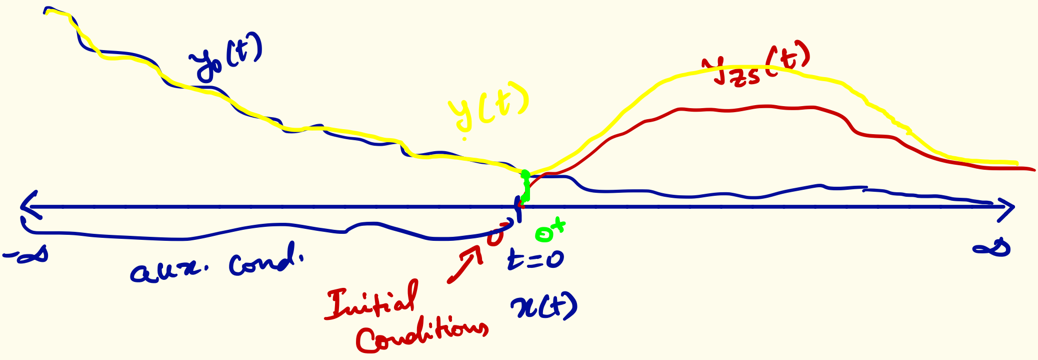
# 4. Resonance



$$y(t) = \overset{\text{zero Input}}{\underbrace{y_0(t)}} + \overset{\text{zero state}}{\underbrace{y_{zs}(t)}} \qquad -\infty<t<\infty$$

Independent

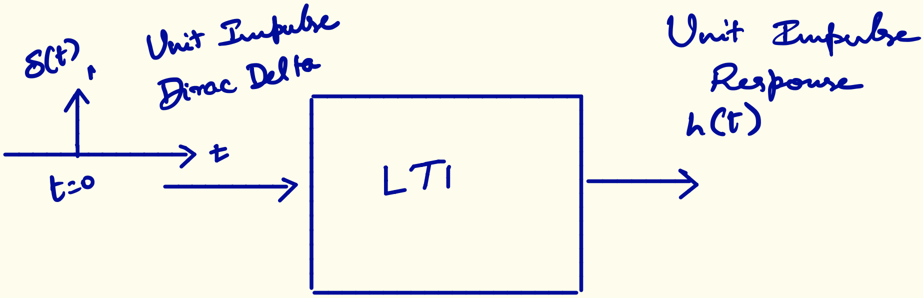
LTIC Systems



$c_1, c_2, \ldots$

$$y_0(0^-) = y_0(0^+)$$

$$y(0^-) \neq y(0^+) \rightarrow y_0(0^+) + y_{zs}(0^+)$$



$$h(t) = A_0 \delta(t) + \text{linear combination of characteristic modes } e^{\lambda_i t}$$ ①

M = N

$$x(t) = \delta(t)$$
$$y(t) = h(t)$$

$$\overbrace{\left(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N\right)}^{Q(D)} h(t)$$

$$= \overbrace{\left(b_0 D^N + b_1 D^{N-1} + \dots + b_{N-1} D + b_N\right)}^{P(D)} \delta(t)$$ ②

from ① & ② $\Rightarrow A_0 D^N \delta(t) = b_0 D^N \delta(t)$

$A_0 = b_0$ for $M = N$

$A_0 = 0$ for $M < N$

# Impulse Response

$Q(D)\,y(t) = P(D)\,x(t)$

linear combn. of char. modes

Unit step

$$h(t) = b_0\,\delta(t) + \big[P(D)\,y_n(t)\big]\,u(t)$$

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = \dots = y_n^{(N-2)}(0) = 0$$

$$y_n^{(N-1)}(0) = 1$$

$M \leq N$

$b_0 = 0,\ M < N$

$N = 1 \qquad y_n(0) = 1$

$N = 2 \qquad y_n(0) = 0, \quad \dot{y}_n(0) = 1$

**eg.**

$$\overset{Q(D)}{(D^2 + 3D + 2)}\,y(t) = \overset{P(D)}{D}\,x(t)$$

$$P(D) = D, \quad b_0 = 0$$

$$h(t) = \underset{\substack{\uparrow \\ d/dt}}{D}\left[c_1 e^{-2t} + c_2 e^{-t}\right] u(t)$$

$$y_n(t) = c_1 e^{-2t} + c_2 e^{-t}$$

$y_n(0) = 0$

$\dot{y}_n(0) = 1$

$$y_n(t) = -e^{-2t} + e^{-t}$$

$$\dot{y}_n(t) = D\,y_n(t) = 2e^{-2t} - e^{-t}$$

$$h(t) = \left[2e^{-2t} - e^{-t}\right] u(t)$$

eg.

$$\overset{Q(D)}{(D^2+4D+4)}y(t) = \overset{P(D)}{(D^2+3D+2)}x(t)$$

$y_n(0)=0$

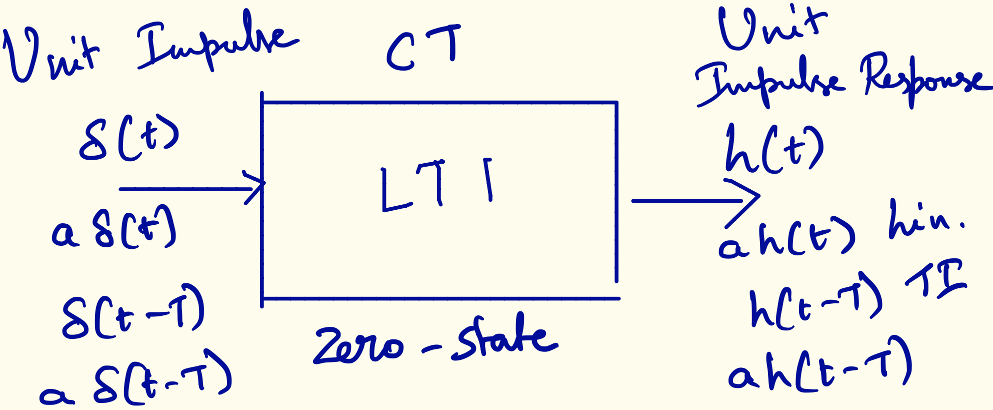
$\dot{y}_n(0)=1$

Impulse Response

$$h(t) = \delta(t) + P(D)(0+1t)e^{-2t}u(t)$$

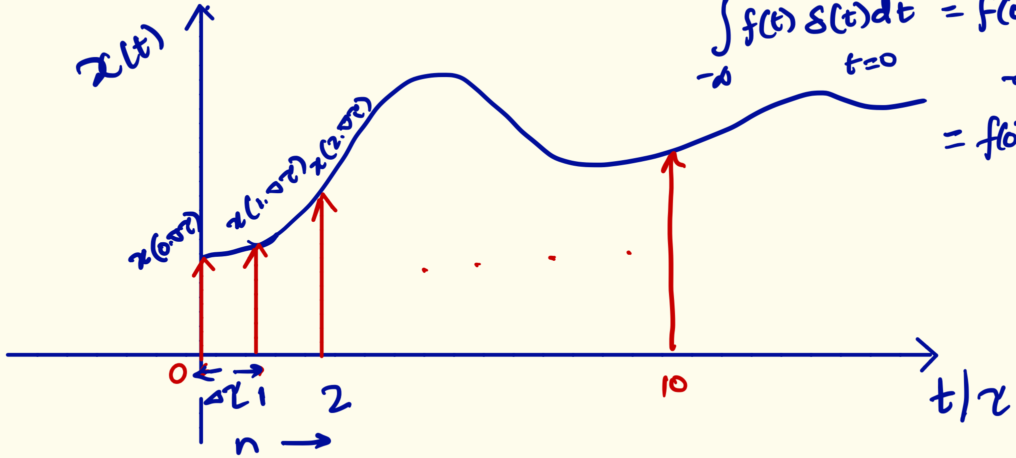
$$= \delta(t) + (D^2+3D+2)te^{-2t}u(t)$$

$$y_n(t) = (c_1+c_2t)e^{-2t}$$



$$\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt \underset{t=0}{} = f(0)\int_{-\infty}^{+\infty}\delta(t)\,dt$$

$$= f(0)$$



$$x(t) = x(0\cdot\Delta\tau)\,\delta(t - 0\cdot\Delta\tau)\,\Delta\tau$$

$$\lim_{\Delta\tau\to 0}\sum_{\tau} x(n\Delta\tau)\,\delta(t-n\Delta\tau)\,\nabla\tau$$

as $\nabla\tau \to \overset{\tau}{0}$, $n\Delta\tau \to \overset{\tau}{0}$

$$x(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau \quad \text{——} \ ①$$

$$x(9) \Rightarrow \int_{-\infty}^{+\infty} x(\tau)\,\delta(9-\tau)\,d\tau = x(9)\int_{-\infty}^{+\infty}\delta(9-\tau)\,d\tau$$

$$= x(9)$$

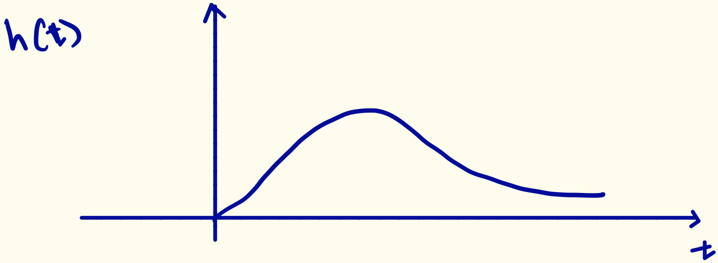
$$x(0) \Rightarrow \int_{-\infty}^{+\infty} x(\tau)\,\delta(-\tau)\,d\tau = x(0)\int_{-\infty}^{+\infty}\delta(\tau)\,d\tau$$

$$= x(0)$$

| Input | LTI<br>$h(t)$ | Output | |
|---|---|---|---|
| $\delta(t)$ | | $h(t)$ | |
| $\delta(t-\tau)$ | | $h(t-\tau)$ | TI |
| $x(\tau)\,\delta(t-\tau)$ | | $x(\tau)\,h(t-\tau)$ | Homo. |
| From ' $x(t)=\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$ | | $y(t)=\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$ | Add. |

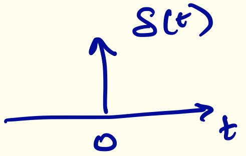
Convolution Integral

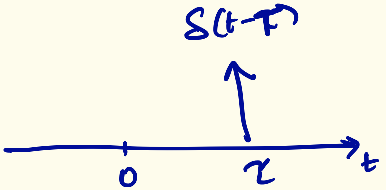
$$y(t) = x(t) * h(t)$$

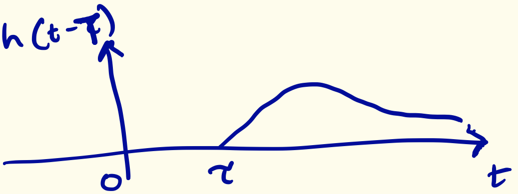


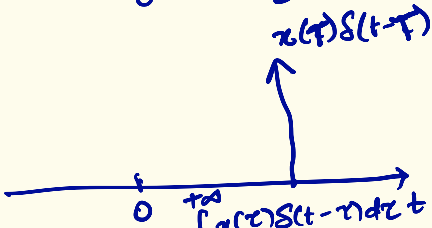
Input

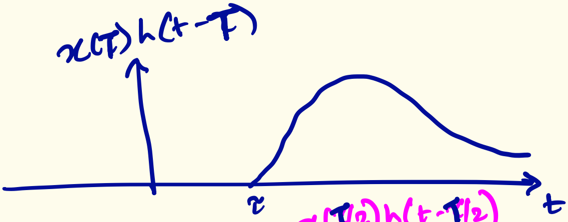
Output





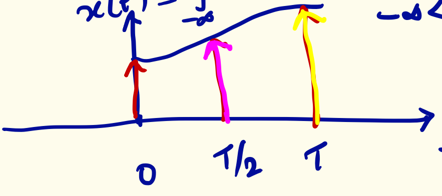


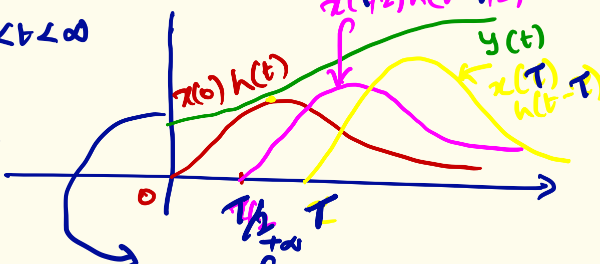




$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

$-\infty < t < \infty$

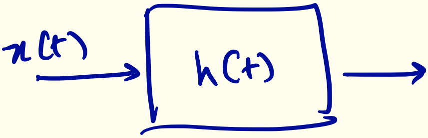




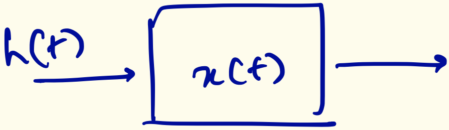
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$-\infty < t < \infty$

# Commutative

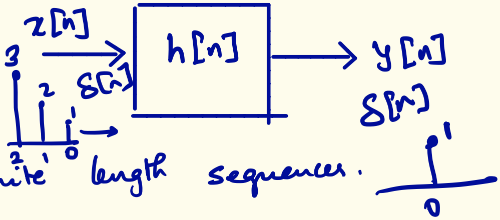


$y(t) = x(t) * h(t)$



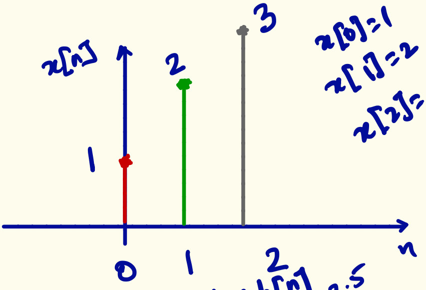
$y(t) = h(t) * x(t)$

# A Discrete Example

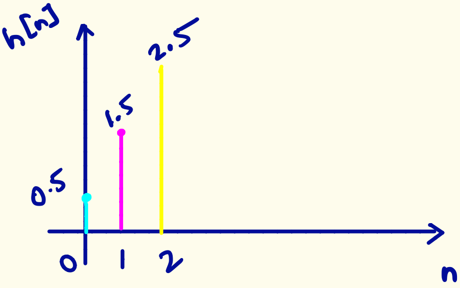


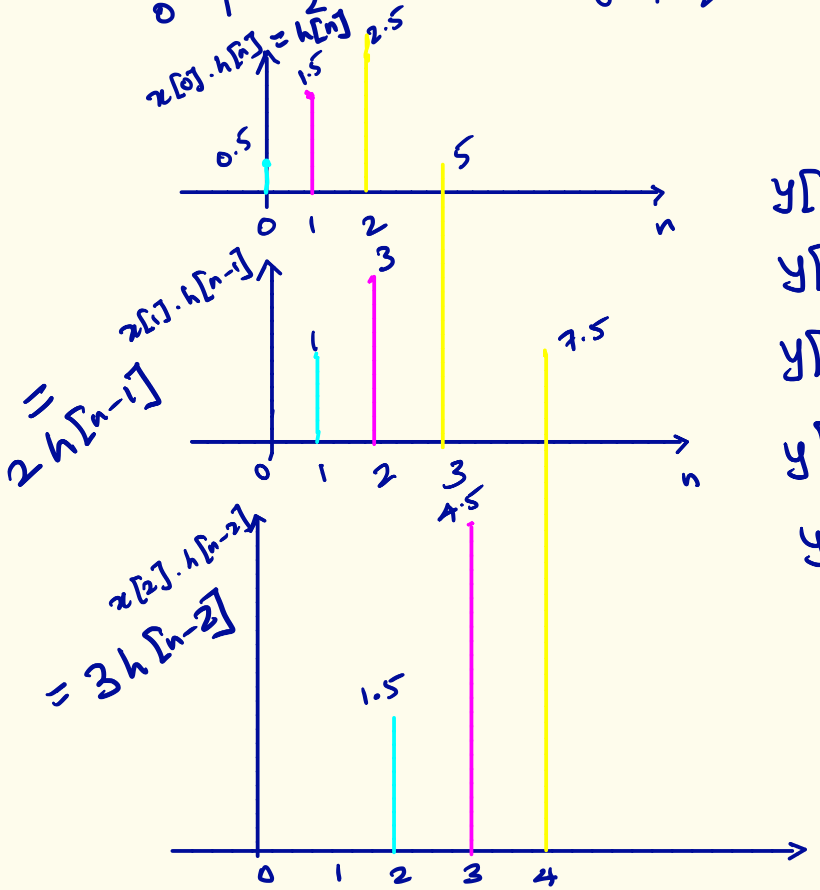
x[n], h[n] → Finite length sequences.

①



x[0]=1
x[1]=2
x[2]=3





y[0] = 0.5
y[1] = 2.5
y[2] = 7
y[3] = 9.5
y[4] = 7.5

②

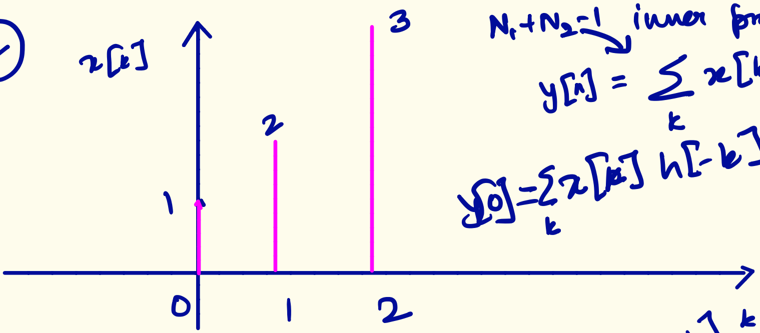
$N_1+N_2-1$ inner products

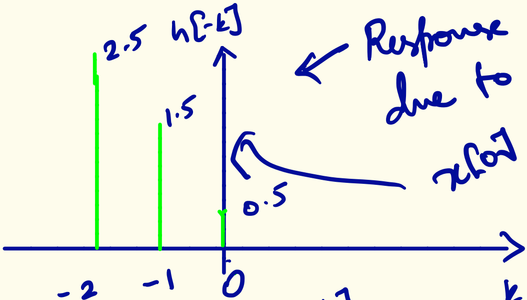
$$y[n] = \sum_k x[k]\, h[n-k]$$

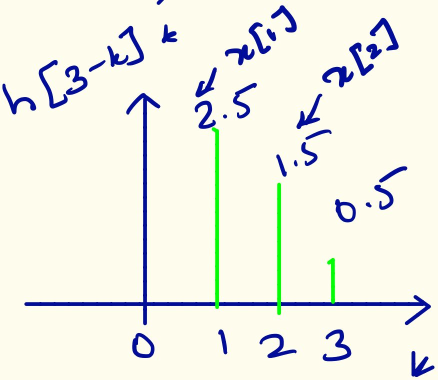
$$y[0] = \sum_k x[k]\, h[-k]$$

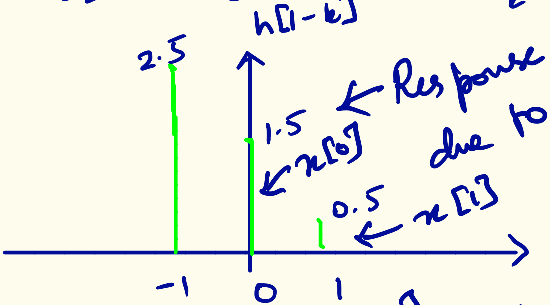
$x[n] \rightarrow N_1$

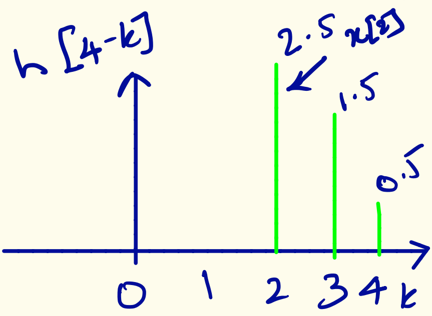
$h[n] \rightarrow N_2$

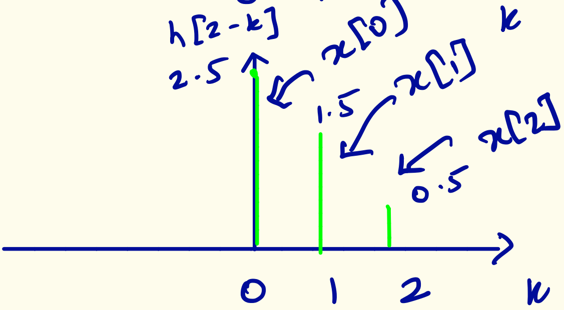












$y[0] = x[k] \cdot h[-k]$
$= 0.5$

$y[1] = x[k] \cdot h[1-k]$
$= 2.5$

$y[n] = N_1+N_2-1$

$y[2] = 7$,

$y[3] = 9.5$

$y[4] = 7.5$

$$x[k] \cdot h[-k] = \sum_{k} x[k]\, h[-k]$$

$$\equiv \int_{\tau} x(\tau)\, h(-\tau)\, d\tau \rightarrow \text{Area under the curve } x(\tau)\,h(-\tau)$$

$$\downarrow$$

$$y(0)$$

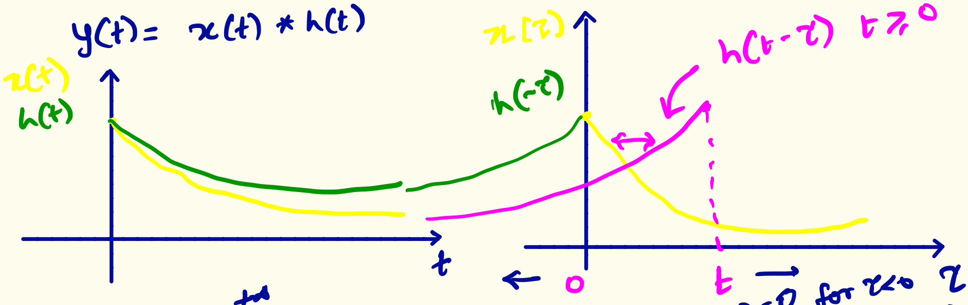
$$y(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

eg. 1 $x(t) = e^{\lambda_1 t} u(t)$ Causal I/P

$h(t) = e^{\lambda_2 t} u(t)$ → LTI Causal

$\lambda_1 < \lambda_2 < 0$

$y(t) = x(t) * h(t)$



for $t<0$, $\int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = 0$ as $x(\tau)=0$ for $\tau<0$

for $t \geq 0$, $\int_{0^-}^{t} x(\tau)\, h(t-\tau)\, d\tau$ as $\begin{cases} x(\tau)=0 \text{ for } \tau<0 \rightarrow 0^- \\ h(t-\tau)=0 \text{ for } \tau>t \rightarrow t \end{cases}$

$$= \int_{0}^{t} e^{\lambda_1 \tau}\, e^{\lambda_2 (t-\tau)}\, d\tau$$

# Dot Product | Inner Product

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}_{1\times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$$

$$= \sum_{i=1}^{n} x_i y_i$$

---

$$y(t) = \int_0^t e^{\lambda_1 \tau} \, e^{\lambda_2 (t-\tau)} \, d\tau$$

$$= e^{\lambda_2 t} \int_0^t e^{(\lambda_1 - \lambda_2)\tau} \, d\tau$$

$$= e^{\lambda_2 t} \left. \frac{e^{(\lambda_1 - \lambda_2)\tau}}{(\lambda_1 - \lambda_2)} \right|_0^t$$

$$= e^{\lambda_2 t} \frac{e^{(\lambda_1 - \lambda_2)t} - 1}{(\lambda_1 - \lambda_2)} = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} u(t), \ \lambda_1 \neq \lambda_2$$

**eg. 2**

$$x(t) = e^{\lambda t} u(t)$$

$$h(t) = e^{\lambda t} u(t)$$

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t e^{\lambda\tau} e^{\lambda(t-\tau)} d\tau$$

$$= \int_0^t e^{\lambda t} d\tau = e^{\lambda t} \int_0^t d\tau$$

$$= e^{\lambda t}\ \tau \Big|_0^t$$

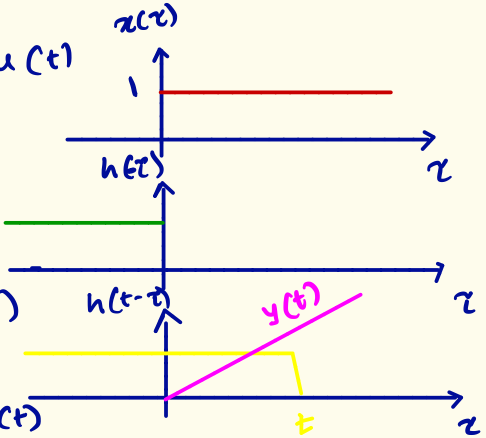
$$y(t) = t e^{\lambda t}\ u(t)$$

**eg. 3**

$$x(t) = u(t)$$

$$h(t) = u(t)$$

$$y(t) = t\, u(t)$$



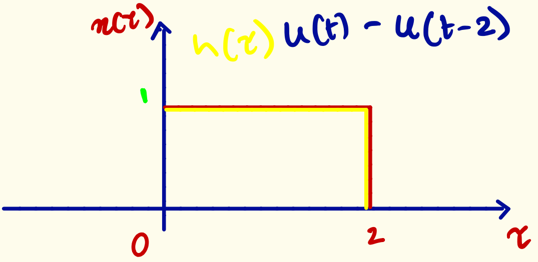
**eg. 4**

$$x(t) = e^{\lambda t} u(t)$$

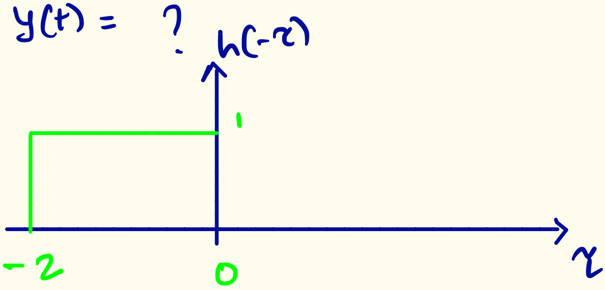
$$h(t) = u(t) \qquad \lambda = 0$$

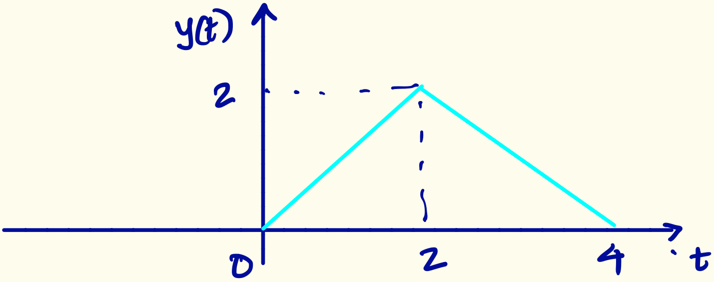
$$y(t) = \frac{e^{\lambda t} - 1}{\lambda}\, u(t)$$

eg. 5.

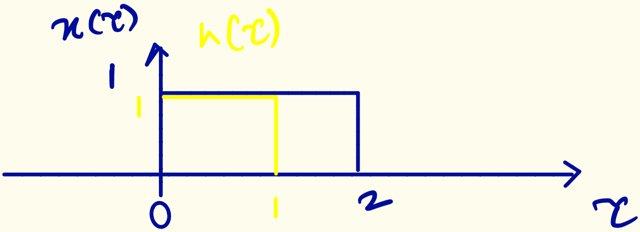


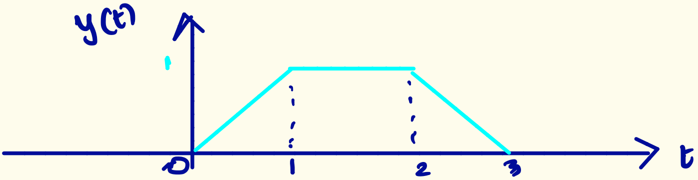
y(t) = ?





eg. 6





# Convolution Steps

Given $x(t)$, $h(t)$

Output $y(t)$

1. Replace $t$ by $\tau$ $\quad x(\tau), h(\tau)$

2. Flip $h(\tau) \rightarrow h(-\tau)$

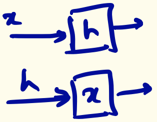
3. Shift $h(-\tau) \longrightarrow h(t-\tau)$

4. Evaluate $y(t) = \int_{\tau} x(\tau)\, h(t-\tau)\, d\tau$

for all valid 't' → there is overlap between $x(\tau)$ & $h(t-\tau)$

# Properties of Convolution

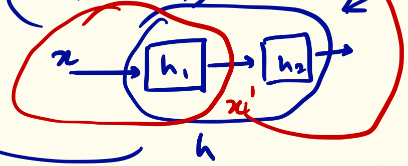
1. Commutative



$$x(t) * h(t) = h(t) * x(t)$$

2. Associative

$$x(t) * \left(h_1(t) * h_2(t)\right) = \left(x(t) * h_1(t)\right) * h_2(t)$$

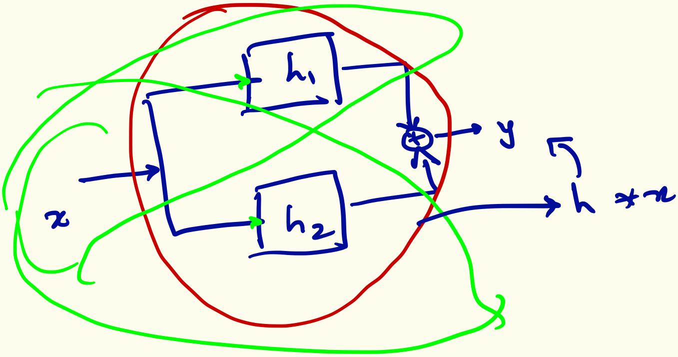


Systems in series

3. Distributive

$$x(t) * \underbrace{\left[h_1(t) + h_2(t)\right]}_{h} = x(t) * h_1(t) + x(t) * h_2(t)$$

Systems in Parallel



4. $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x * h$

$x(t-T) \longrightarrow \boxed{h(t)} \longrightarrow y(t-T)$

$x(t-T_1) \longrightarrow \boxed{h(t-T_2)} \longrightarrow y(t-T_1-T_2)$

$x(t-T_1) \longrightarrow \boxed{h(t+T_1)} \longrightarrow y(t)$

5. $x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

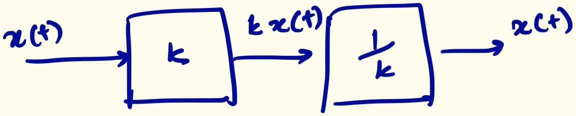
$$= x(t)$$

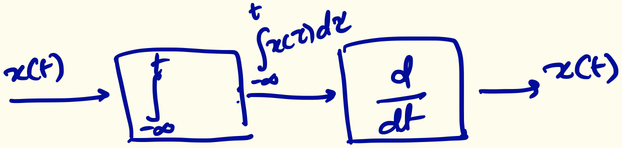
6. Inverse System

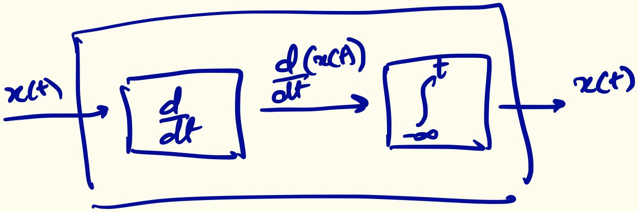
$x(t) \longrightarrow \boxed{h_1(t)} \xrightarrow{y_1(t)} \boxed{h_2(t)} \longrightarrow y_2(t) = x(t)$ $\qquad h_2 = h_1^{-1}$

eg.

## Cascade





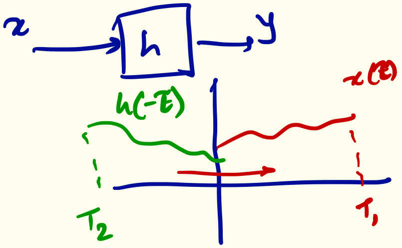


## 7. Width

width

$x(t) \longrightarrow T_1$

$h(t) \longrightarrow T_2$



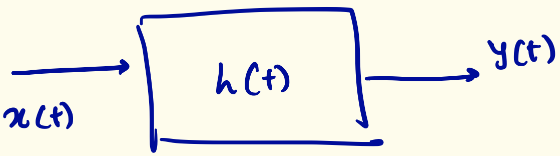
$y(t) = x(t) * h(t) \longrightarrow T_1 + T_2$

## Total Response of LTI Continuous Time System

zero input  zero state

$$y(t) = y_0(t) + y_{zs}(t)$$

$$\sum c_i e^{\lambda_i t} + x(t) * h(t)$$



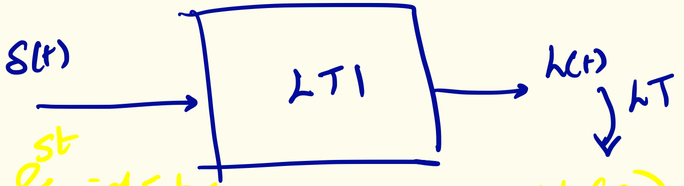
$$y(t) = y_0(t) + y_{zs}(t)$$

$$= \underbrace{2e^{-3t} + e^{-5t}} + x(t) * h(t)$$

$$= \underbrace{2e^{-3t} + e^{-5t}}_{y_0(t)} + \underbrace{\overset{\text{New}}{e^{-t}} - 3e^{-3t} + \overset{\text{New}}{e^{-4t}}}_{y_{zs}(t)}$$

$$= \underbrace{-e^{-3t} + e^{-5t}}_{\text{Natural Response}} + \underbrace{e^{-t} + e^{-4t}}_{\text{Forced Response}}$$

# Zero-state



$e^{st}$, $-\infty < t < \infty$

$s = \sigma + j\omega$

Everlasting Exponential

$$y(t) = H(s)\, e^{st}$$

ZS

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$x(t) = e^{st}$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$= (b_0 D^M + b_1 D^{M-1} + \dots + b_M)\, e^{st}$$

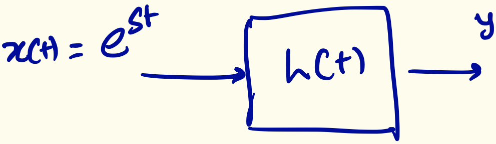
$D(e^{st}) = s e^{st}$

$D^M(e^{st}) = s^M e^{st}$

$$= (b_0 s^M + b_1 s^{M-1} + \dots + b_M)\, e^{st}$$

$$= P(s)\, e^{st} \longrightarrow ②$$

$$Q(D)\, y(t) = (D^N + a_1 D^{N-1} + \dots + a_N)\, y(t)$$



$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{+\infty} e^{s\tau}\, h(t-\tau)\, d\tau$$

Comm.

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$y(t) = e^{st} H(s) \longrightarrow ①$$

$$y(t) = H(s)\, e^{st} \qquad \text{————} \textcircled{1}$$

$$Q(D)\, y(t) = P(s)\, e^{st} \qquad \text{————} \textcircled{2}$$

① in ②

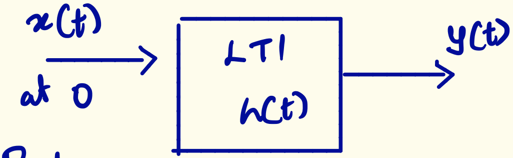
$$D = \frac{d}{dt}$$

$$Q(D)\, \underline{H(s)}\, e^{st} = P(s)\, e^{st}$$

$$H(s)\, Q(D)\, e^{st} = P(s)\, e^{st}$$

$$H(s)\, Q(s)\, \cancel{e^{st}} = P(s)\, \cancel{e^{st}}$$

$$H(s) = \left.\frac{P(s)}{Q(s)}\right|_{x(t)=e^{st}} \longrightarrow \text{Transfer Function of the System}$$



Total Response

$$y(t) = \overset{\text{zero input}}{y_0(t)} + \overset{\text{zero state}}{y_{zs}(t)}$$

$$= \sum \underset{\underset{t \leq 0^-}{\uparrow}}{c_i}\, e^{\lambda_i t} + x(t) * h(t) \longrightarrow b_0\delta(t) + P(D)\overset{\overset{\sum c_i' e^{\lambda_i t}}{\uparrow}}{y_n(t)}\, u(t)$$

$$= \overset{\text{Natural}}{\hat{y}(t)} + \overset{\text{Forced}}{y_\phi(t)}$$

$$= \sum \underset{\underset{t \geq 0^+}{\uparrow}}{c_i''}\, e^{\lambda_i t} + y_\phi(t)$$

$$\underline{Q(D)}\, y(t) = P(D)\, x(t)$$

$\downarrow$

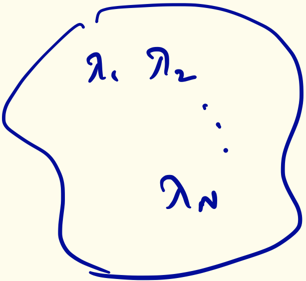
roots → char. values

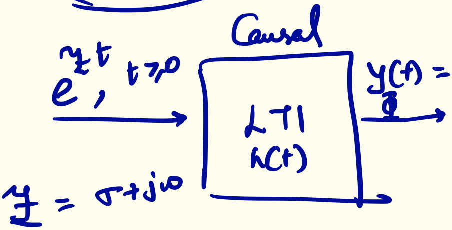
Zero input → characteristic Modes { Soln of $Q(D)=0$

Zero state → characteristic Modes + Forced Modes

Natural → characteristic Modes

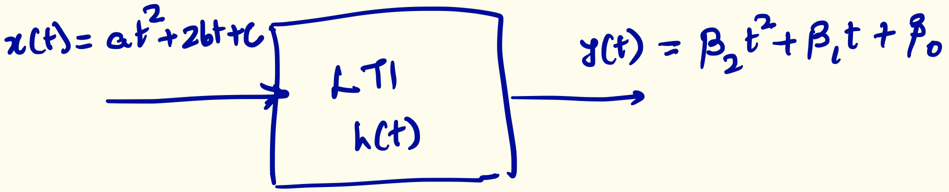
Forced → Forced Modes





$y(t) = H(z)\, e^{zt} \quad t \geq 0$

$$H(z) = \frac{P(z)}{Q(z)}$$



Case 1 $Ce^{zt}$, $z = \sigma + j\omega$ $t \geq 0$

$\sigma = 0$, $\omega = 0$

$$x(t) = Ce^{0t} \quad t \geq 0$$

$$y_{\phi}(t) = H(z)\,Ce^{0t} \quad t \geq 0$$

$$= C\,H(0)$$

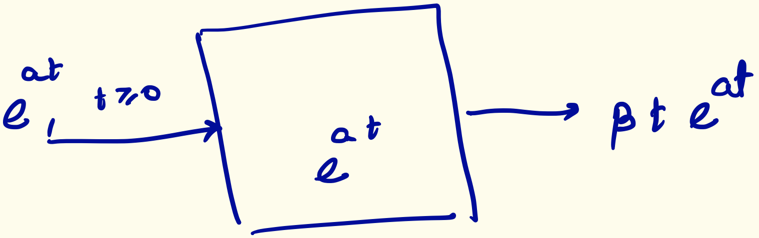
$$= C\,\frac{P(0)}{Q(0)}$$

$$Q(D)\,y(t) = P(D)\,x(t)$$

$$(D^2+3D+2)\,y(t) = 3D\,x(t)$$

$$= (0D^2+3D+0)\,x(t)$$

## Case 2



## Case 3

$$x(t) = e^{j\omega t}, \quad t \geq 0 \quad \sigma = 0$$

$$y(t) = H(j\omega)\, e^{j\omega t} \qquad t \geq 0$$

## Case 4

$$x(t) = \cos \omega t, \quad t \geq 0$$

$$= \frac{1}{2}\left[ e^{j\omega t} + e^{-j\omega t} \right], \quad t \geq 0$$

$$y(t) = \frac{1}{2}\left[ H(j\omega)\, e^{j\omega t} + H(-j\omega)\, e^{-j\omega t} \right] \; t \geq 0$$

$$= |H(j\omega)| \cos\left(\omega t + \angle H(j\omega)\right)$$

eg:

$$\left(D^2+3D+2\right)y(t) = (D+3)\,x(t)$$

$$y(0^+) = 3$$

$$\dot{y}(0^+) = 5$$

$y(t)$ for $x(t) = e^{-4t}u(t)$.

$$y(t) = \hat{y}(t) + y_\phi(t) \rightarrow H(s)e^{st}u(t)$$

$$= c_1 e^{-t} + c_2 e^{-2t} + H(-4)\,e^{-4t}u(t)$$

$$H(s) = \frac{s+3}{s^2+3s+2}$$

$$y(t) = \left[c_1 e^{-t} + c_2 e^{-2t} - \frac{1}{6}e^{-4t}\right]u(t)$$

# Stability

System → LTI Causal

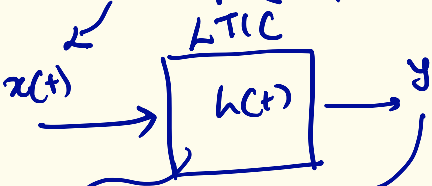
2. zero-input for any initial conditions (internal)

1. zero-state for any input $x(t)$ (external)

Bounded Input Bounded Output (BIBO)

$$|x(t)| \leq K_1 < \infty \quad \forall t$$

Convolution



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \right|$$

$$\leq \boxed{\int_{-\infty}^{+\infty} |x(\tau)|\, |h(t-\tau)|\, d\tau}$$

$$\leq K_1 \int_{-\infty}^{+\infty} |h(t-\tau)|\, d\tau$$

$$|y(t)| \leq K_2 < \infty \quad \forall t$$

Then BIBO stable

$$\int_{-\infty}^{+\infty} |h(\underline{t}-\tau)|\, d\tau \leq K_1' < \infty$$

$$K_2 = K_1 K_1' < \infty$$

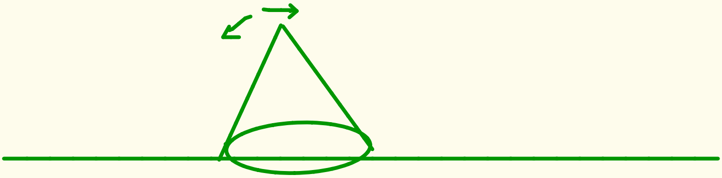
For BIBO stability,

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$$

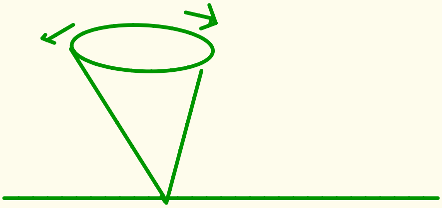
char. modes.

$\searrow$ $b_0\delta(\tau) + P(D)\, y_n(\tau)\, u(\tau)$

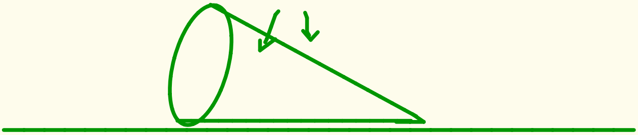
1. Stable



2. Unstable



3. Marginally stable



## 2. Asymptotic Stability (zero input)

$$Q(D)\, y_0(t) = 0$$

↓

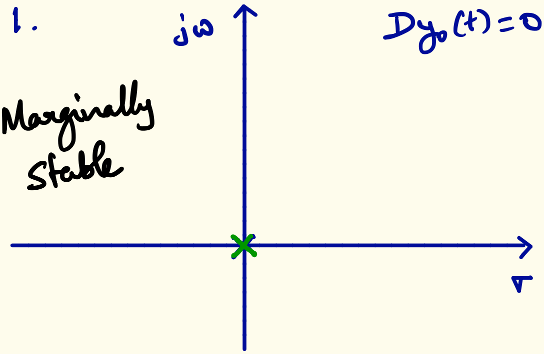
Roots where they lie.

Location of roots

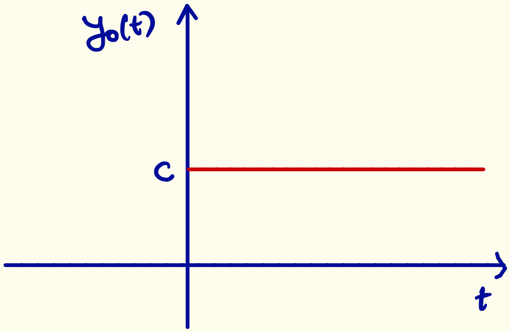
zero-input response

1.

Marginally stable

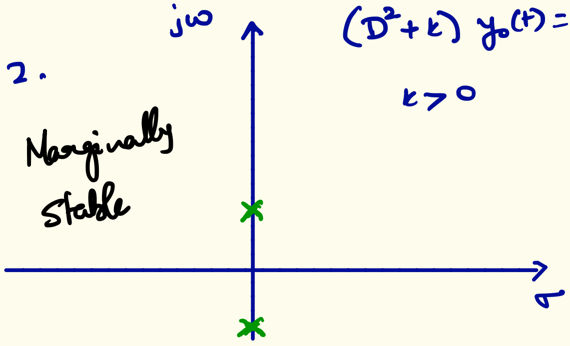


$D y_0(t) = 0$



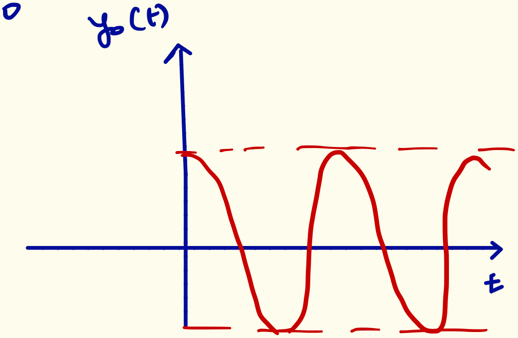
2.

Marginally stable



$(D^2 + k)\, y_0(t) = 0$

$k > 0$



$$y_0(t) = \cos(\omega t)$$

## Case 1

$Q(\lambda)$ has distinct roots.

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_N e^{\lambda_N t}$$

## Case 2

$Q(\lambda)$ has one root with 'r' repetitions.

$$y_0(t) = \left(c_1 + c_2 t + \dots + c_r t^{r-1}\right) e^{\lambda_r t} + c_{r+1} e^{\lambda_{r+1} t}$$
$$+ \dots + c_N e^{\lambda_N t}$$

## Case 3

$Q(\lambda)$ has one pair of complex roots.

Complex conjugate roots $\alpha + j\beta$, $\alpha - j\beta$

$$y_0(t) = \underbrace{c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}}_{} + \dots + c_N e^{\lambda_N t}$$

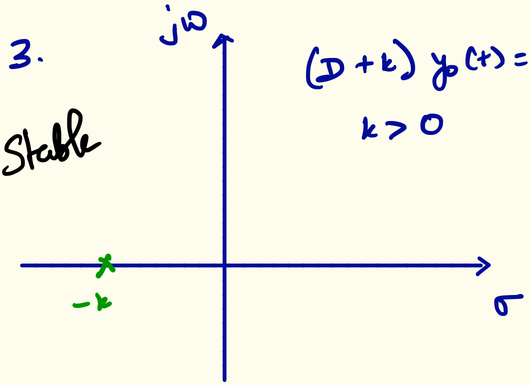
$$e^{\alpha t}\left[c_1 e^{j\beta t} + c_2 e^{-j\beta t}\right]$$
$$= \frac{c}{2} e^{\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right]$$
$$= c e^{\alpha t}\left[\cos(\beta t + \theta)\right]$$

$$c_1 = \frac{c}{2} e^{j\theta}$$

$$c_2 = \frac{c}{2} e^{-j\theta}$$

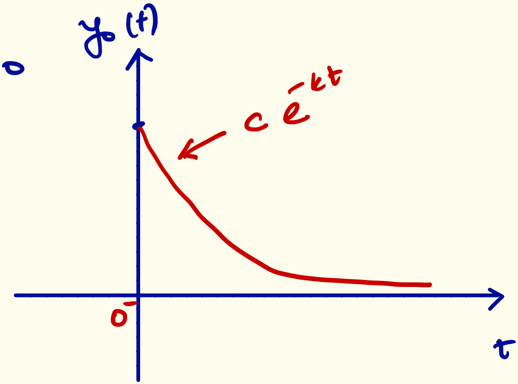
3.

Stable



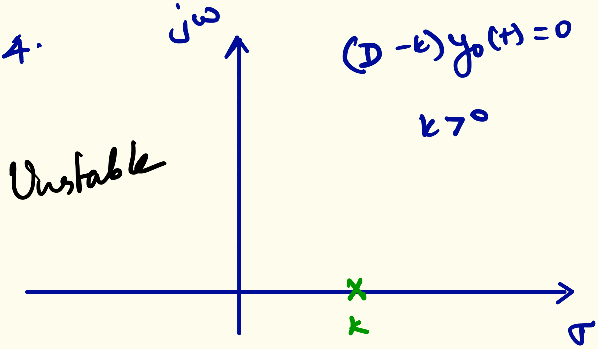
$(D+k)\,y_0(t)=0$

$k>0$



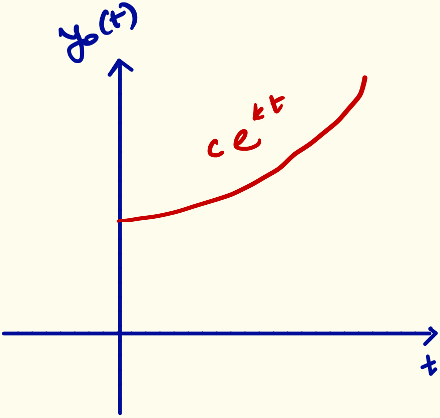
4.

Unstable



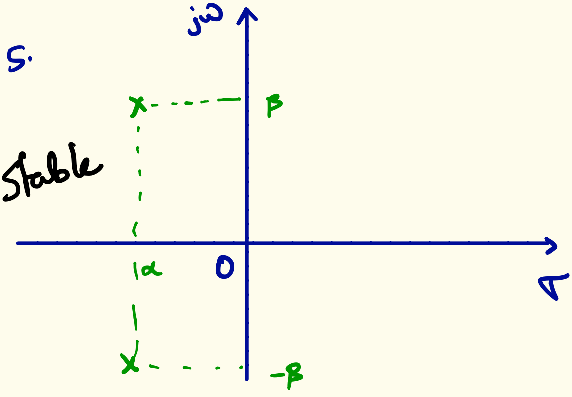
$(D-k)\,y_0(t)=0$

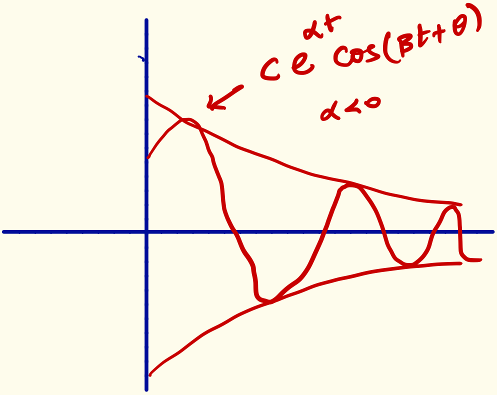
$k>0$



5.

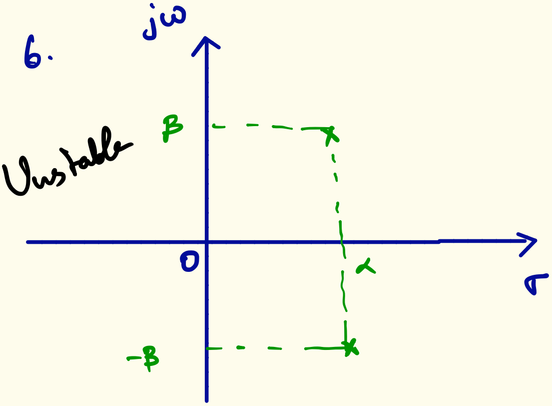
Stable

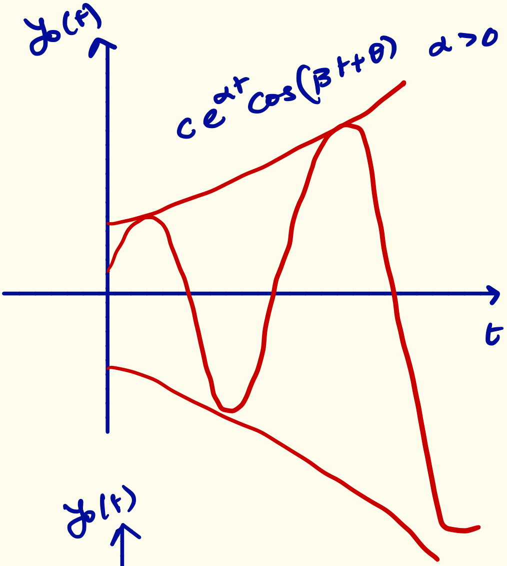




6.

Unstable

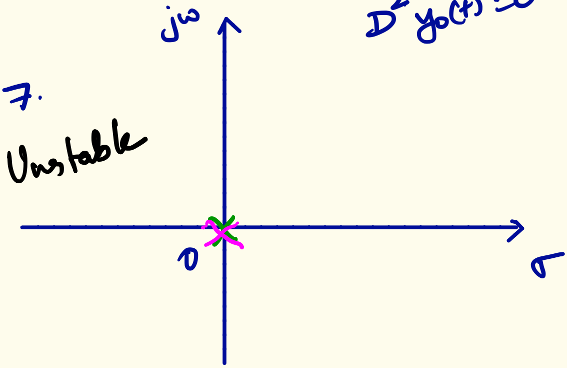


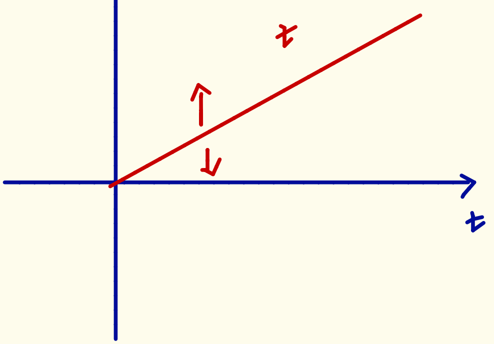


7.

Unstable

$D^2 y_0(t) = 0$

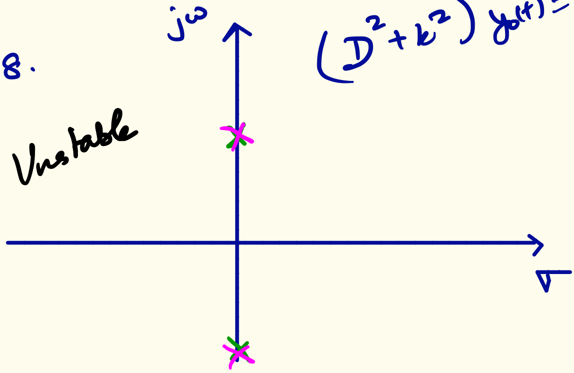


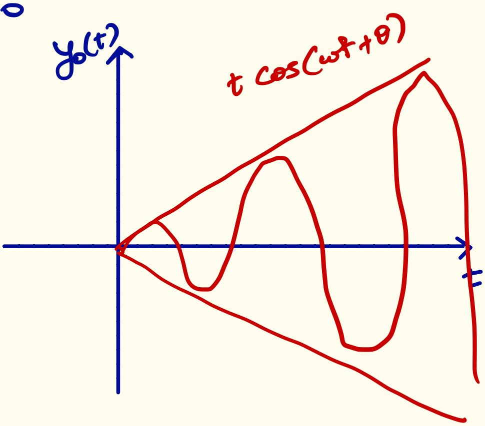


8.

Unstable

$(D^2+k^2)^2 y_0(t) = 0$





## Asymptotic (zero-input) stability

$y_0(t) = \sum c_i e^{\lambda_i t}$

An LTI Causal system is

a) stable if all the characteristic roots lie on LHS of the $(\sigma + j\omega)$ plane (Real part is negative)

b) Marginally stable if some of the char. roots are simple on $j\omega$ axis and all other char. roots lie on LHS of the $(\sigma + j\omega)$ plane.

c) Unstable if

i) some or all of the char. roots lie on the RHS of the $(\sigma + j\omega)$ plane (Real part is positive).

ii) some or all of the char. roots are repeated on the $j\omega$ axis

and the other char. roots are on the LHS of the $(\sigma + j\omega)$ plane

1. Zero-state stability (BIBO)

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty \longrightarrow \text{BIBO stable}$$

char. modes $\longrightarrow 0$ as $t \to \infty$

when all char. roots lie on LHS of $(\sigma + j\omega)$ plane.

$$\begin{tikzcd}
\text{Asymptotically stable} \arrow[r, Rightarrow, shift left] & \text{BIBO stable} \arrow[l, Leftarrow, shift left]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Asymptotically Unstable} \arrow[r, Rightarrow] & \text{BIBO Unstable} \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\text{Asymptotically Marginally} \\ \text{stable}\end{array} \arrow[r, Rightarrow] & \text{BIBO Unstable} \arrow[l, bend left]
\end{tikzcd}$$

$$y_0(t) = \sum c_i e^{\lambda_i t}$$

$$h(t) = b_0 \delta(t) + P(D) \sum c_i' e^{\lambda_i t} u(t)$$

# Check for stability of LTIC system

1. Asymptotic (zero-input)

roots of $Q(\lambda)$ → Stable

roots of $Q(\lambda)$ → Marginally stable

roots of $Q(\lambda)$ → Unstable

2. BIBO (zero-state)

Marginally stable, Unstable → BIBO unstable

Stable → BIBO stable

$$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty \Rightarrow \text{BIBO stable}$$

eg.

1. $D^2(D-3)\,y(t) = (D+3)\,x(t)$

Asymptotically Unstable, BIBO unstable

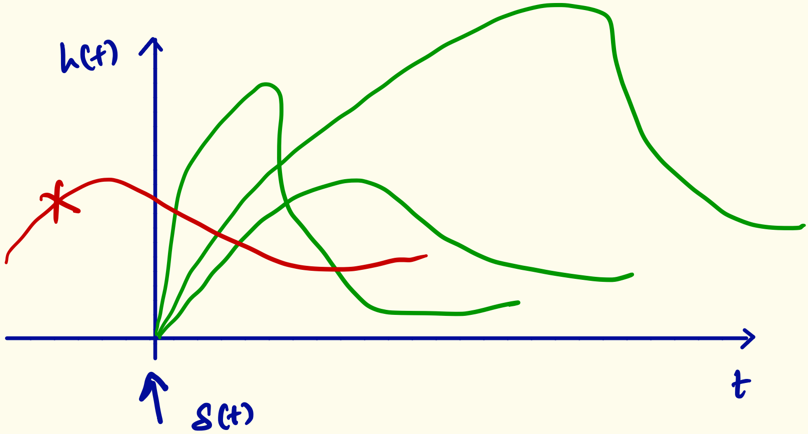
2. $D(D^2+3D+2)\,y(t) = (D+4)\,x(t)$

Marginally stable, BIBO unstable

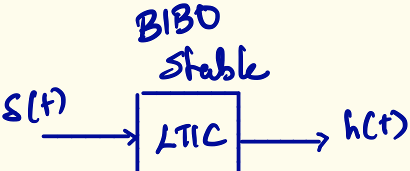
3. $(D^2+2)\,y(t) = D\,x(t)$

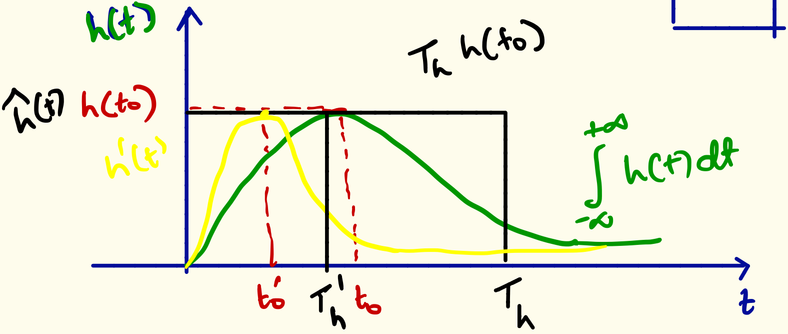
Marginally stable, BIBO unstable

# LTI Causal Systems



# Response Time





Ideal

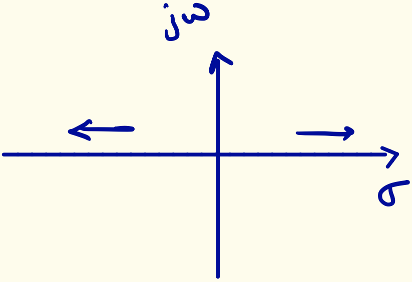
$T_h \rightarrow \infty$

Area under $\hat{h}(t)$ = Area under $h(t)$

$$T_h\, h(t_0) = \int_{-\infty}^{+\infty} h(t)\,dt$$

Practical $T_h = \frac{1}{h(t_0)} \int_{-\infty}^{+\infty} h(t)\,dt$

- Response Time
- Rise Time
- Time Constant

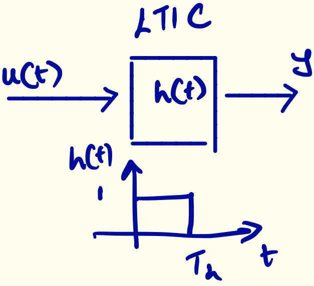


eg. $h(t) = e^{-\lambda t} u(t)$

$$T_h = 1 \cdot \int_{-\infty}^{+\infty} h(t)\,dt = \int_0^{\infty} e^{-\lambda t}\,dt = \frac{1}{\lambda}$$

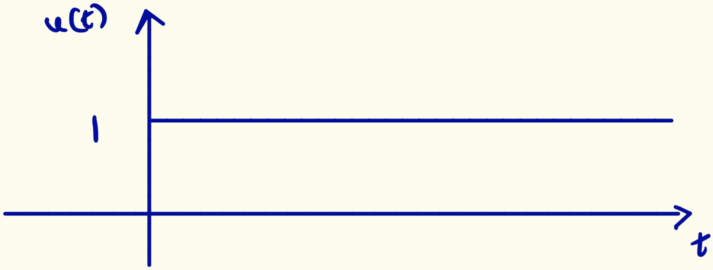
$$\text{time constant} = \frac{1}{\lambda} = \frac{1}{\text{char. root}}$$

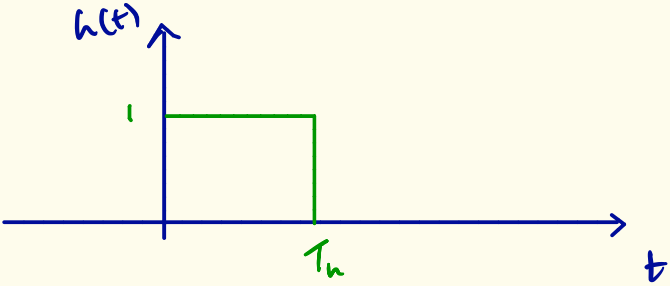
## Step Response

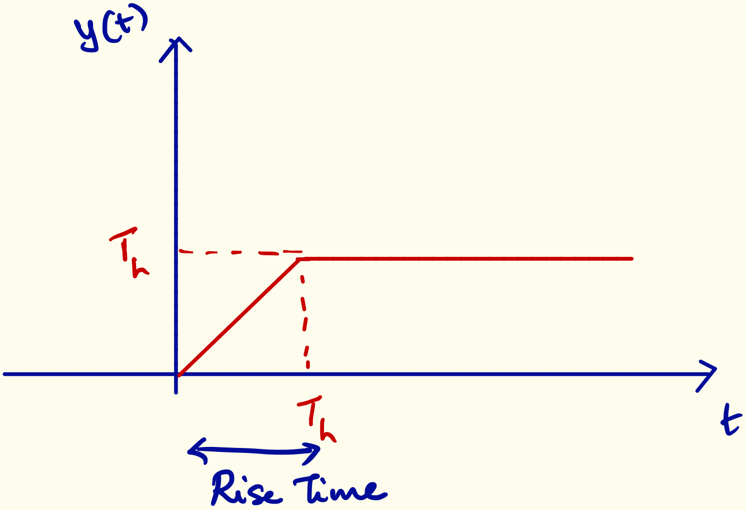


$y(t) \rightarrow$ step response

$= u(t) * h(t)$

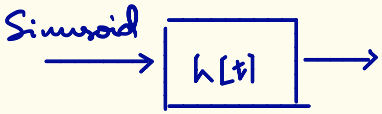


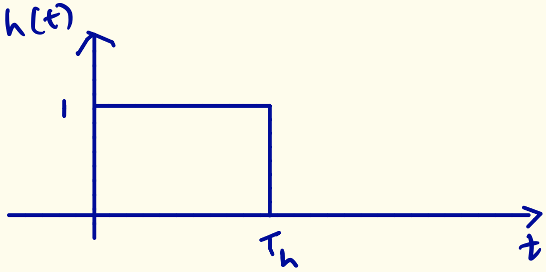




time Taken for output to raise from 10% to 90% of the steady state value.

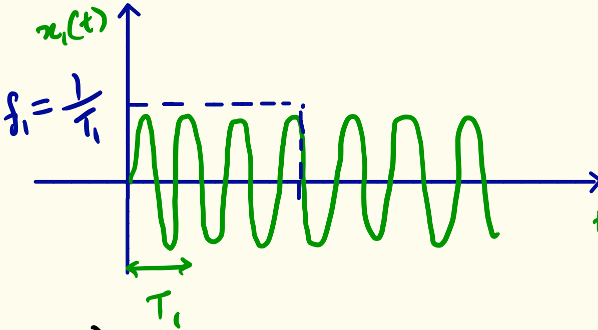
# Filtering

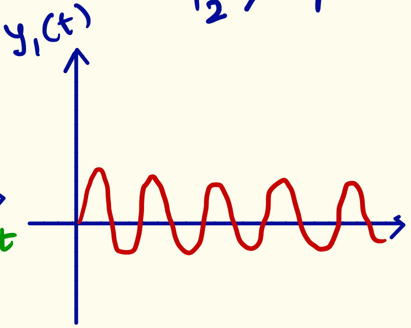


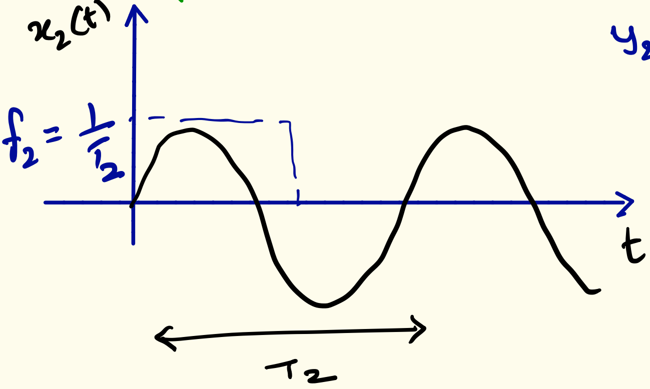


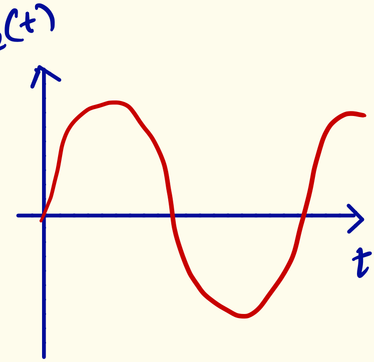
$f_2 < f_1$

$T_2 > T_1$









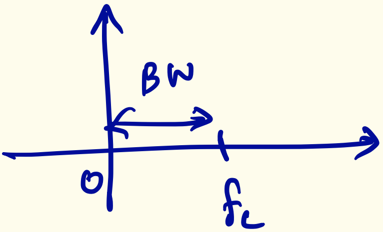
For some $f'$ s.t $f_2 < f' < f_1$

$f' = \frac{1}{T'}$, where $T' = T_h$ or $f' = \frac{1}{T_h}$

$f_c = f' \rightarrow$ cut off freq.

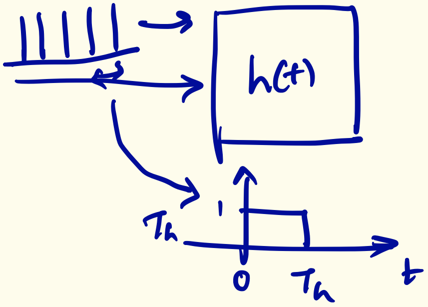
$f_c = \frac{1}{T_h} \rightarrow$ BW / Cut off freq of LPF

$= \frac{1}{\text{Resp. time}}$



**Information rate**

$\propto$ BW of the system



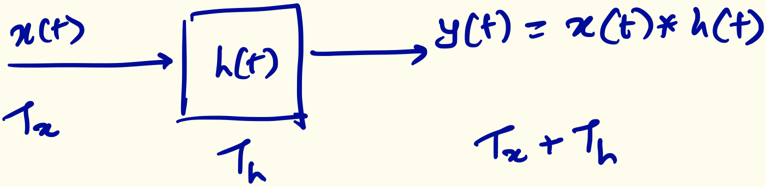
Spacing b/w pulses

$\propto T_h$

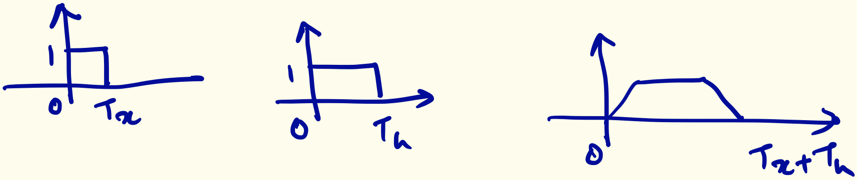
Rate of pulses
No. per sec.

$\propto BW = \frac{1}{T_h}$

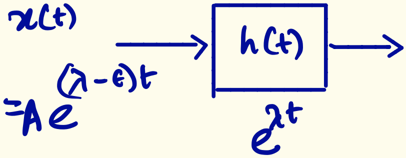
# Pulse dispersion



$$y(t) = x(t) * h(t)$$



# Resonance



$$y(t) = A \frac{1}{\epsilon}\left[e^{\lambda t} - e^{(\lambda-\epsilon)t}\right]$$

$$= \frac{Ae^{\lambda t}\left[1 - e^{-\epsilon t}\right]}{\epsilon}$$

$$\lim_{\epsilon \to 0} \frac{\left[1 - e^{-\epsilon t}\right]}{\epsilon} = \lim_{\epsilon \to 0} \frac{te^{-\epsilon t}}{1} = t$$

$$y(t) = Ate^{\lambda t}$$

$$\lambda = \pm j\omega$$

$$y(t) = At\cos\omega t$$

# Laplace Transform

- Laplace
- Heaviside

For a signal $x(t)$, Bilateral LT

LT $$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad s = \sigma + j\omega$$

ILT $$x(t) = \int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds$$

**eg 1** $$x_1(t) = e^{-at} u(t) \leftarrow 0 \text{ to } \infty \qquad a>0$$

$$X(s) = \int_0^{\infty} e^{-at} e^{-st}\, dt = \int_0^{\infty} e^{-(s+a)t}\, dt$$

$$= \left. \frac{e^{-\overset{+ve}{\underline{(s+a)}}t}}{-(s+a)} \right|_0^{\infty} = \overset{\swarrow}{0} - \left[ -\frac{1}{s+a} \right] = \frac{1}{s+a}$$

$$e^{-(s+a)t} = e^{-\underline{(\sigma+a)}t} \boxed{e^{-j\omega t}} \rightarrow 0$$

$$Re(s+a) > 0$$

$$Re(s) > -a \quad ROC$$

eg 2

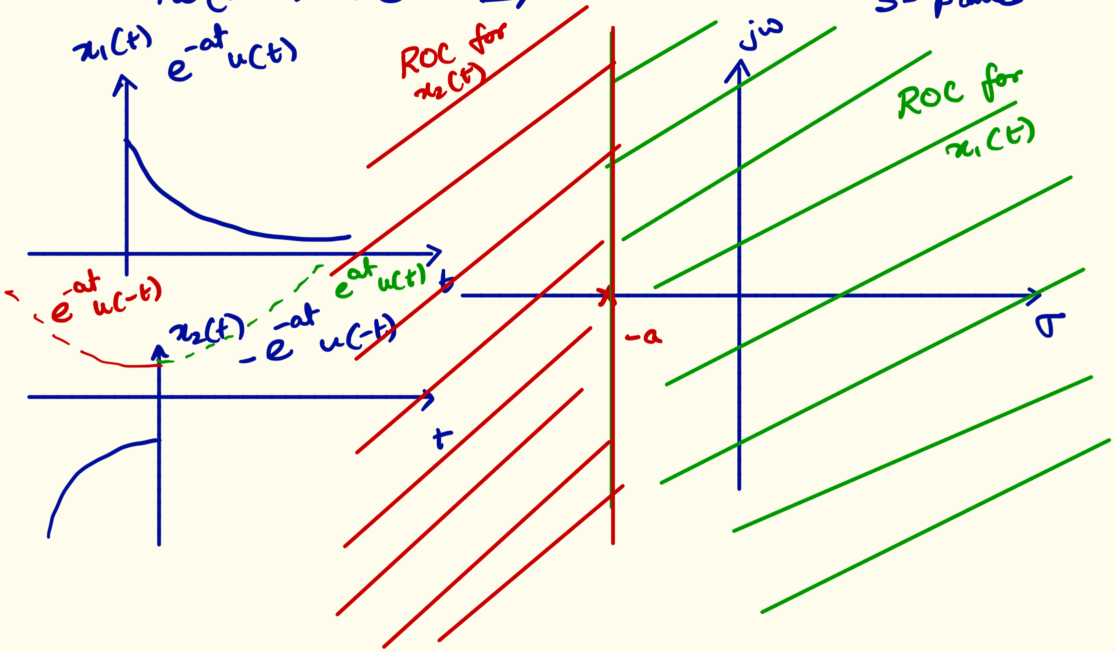
$$x_2(t) = -e^{-at} u(-t) \quad a>0$$

$$X(s) = -\int_{-\infty}^{0} e^{-at} e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}\, dt = -\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{-\infty}^{0}$$

$$= \frac{1}{s+a}\left[e^{-(s+a)t}\right]_{-\infty}^{0} = \frac{1}{s+a}[1-0] = \frac{1}{s+a}$$

$$Re(s+a) < 0 \implies Re(s) < -a$$



## Unilateral Laplace Transform

$$X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

- $x(t)$ is causal signal (impulse response)

## Existence

$$X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

$$|X(s)| < \infty$$

$$\Rightarrow \left| \int_{0^-}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt \right| < \infty$$

$$\hookrightarrow \; < \int_{0^-}^{\infty} |x(t)\, e^{-\sigma t}| \underset{\overset{\shortparallel}{1}}{|e^{-j\omega t}|}\, dt$$

$$< \int_{0^-}^{\infty} |x(t)\, e^{-\sigma t}|\, dt$$

Find when

$$\int_{0^-}^{\infty} \left| x(t) \underline{e^{-\sigma t}} \right| dt < \infty$$

$$x(t) \leq M e^{\sigma_0 t} \qquad \sigma_0 > 0$$

$$\int_{0^-}^{\infty} \left| M e^{\sigma_0 t} e^{-\sigma t} \right| dt \overset{?}{<} \infty$$

$$\sigma_0 - \sigma < 0$$

$$\sigma_0 < \sigma$$

for eg1. $\sigma_0 = -a$ $\quad Re(s) > -a$

## Properties

$$x_1(t) \longrightarrow \boxed{\mathcal{L}} \longrightarrow X_1(s)$$

LT is linear

$x_2(t)$ $\qquad X_2(s)$

$a_1 x_1(t) + a_2 x_2(t)$ $\qquad a_1 X_1(s) + a_2 X_2(s)$

For finite duration signal $x_f(t)$, LT $X_f(s)$ always exists with ROC → entire s-plane.

1.

$$x(t)u(t) \longrightarrow \boxed{\mathcal{L}} \longrightarrow X(s)$$

$x(t-t_0)\,u(t-t_0),\ t_0>0$ $\qquad\qquad$ $e^{-st_0}X(s)$

$$\longrightarrow \boxed{\mathcal{L}} \longrightarrow$$

$$\int_{0^-}^{\infty} x(t-t_0)\,u(t-t_0)\,e^{-st}\,dt$$

$t - t_0 = t'$

$dt = dt'$

$t=0$

$t' = -t_0$

$t=\infty$

$t' = \infty$

$$= \int_{\underline{-t_0}}^{\infty} x(t')\,\underline{u(t')}\,e^{-s(t'+t_0)}\,dt'$$

$$= e^{-st_0}\int_{0^-}^{\infty} x(t')\,e^{-st'}\,dt'$$

$$= e^{-st_0}\,X(s)$$

2. $\xrightarrow{x(t)} \boxed{LT} \longrightarrow X(s)$

$\xrightarrow{x(t)e^{s_0 t}} \boxed{LT} \longrightarrow X(s-s_0)$

$$\int_{0^-}^{\infty} x(t)\, e^{s_0 t}\, e^{-st}\, dt = \int_{0^-}^{\infty} x(t)\, e^{-(s-s_0)t}\, dt$$

$$= X(s-s_0)$$

3. $\xrightarrow{x(t)} \boxed{LT} \xrightarrow{X(s)}$

$\xrightarrow{\frac{d}{dt}x(t)} \boxed{LT} \longrightarrow sX(s) - x(0^-)$

$$\int_{0^-}^{\infty} \frac{d}{dt}\underbrace{x(t)}_{v}\, \underset{u}{e^{-st}}\, dt = e^{-st}x(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} x(t)(-s)\,e^{-st}\,dt$$

$$= -x(0^-) + s\int_{0^-}^{\infty} x(t)\,e^{-st}\,dt$$

$$= sX(s) - x(0^-)$$

$$\int_{0^-}^{\infty} \frac{d^2}{dt^2} x(t) \, e^{-st} dt$$

$$= \int_{0^-}^{\infty} \frac{d}{dt} \left[ \overbrace{\frac{d}{dt} x(t)}^{\dot{x}(t)} \right] e^{-st} dt$$

$$= s \mathcal{L} \left[ \dot{x}(t) \right] - \dot{x}(0^-)$$

$$= s \left[ s X(s) - x(0^-) \right] - \dot{x}(0^-)$$

$$= s^2 X(s) - s \, x(0^-) - \dot{x}(0^-)$$

$$\frac{d^n}{dt^n} x(t) \xrightarrow{LT} s^n X(s) - s^{n-1} x(0^-) - s^{n-2} \dot{x}(0^-) \dots - x^{n-1}(0^-)$$

## 4. Integration

$$x(t) \longrightarrow \boxed{\mathcal{L}} \longrightarrow X(s)$$

$$\int_{0^-}^{t} x(\tau)\,d\tau \longrightarrow \boxed{\mathcal{L}} \longrightarrow \frac{X(s)}{s}$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xrightarrow{\mathcal{L}} \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s}$$

## 5. Convolution

$$x_1(t) \xrightarrow{\mathcal{L}} X_1(s)$$

$$x_2(t) \xrightarrow{\mathcal{L}} X_2(s)$$

$$x_1(t) * x_2(t) \xrightarrow{\mathcal{L}} ?$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\mathcal{L}\left(x_1(t) * x_2(t)\right) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\, e^{-st}\, dt$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t')\, e^{-s(t'+\tau)}\, d\tau\, dt'$$

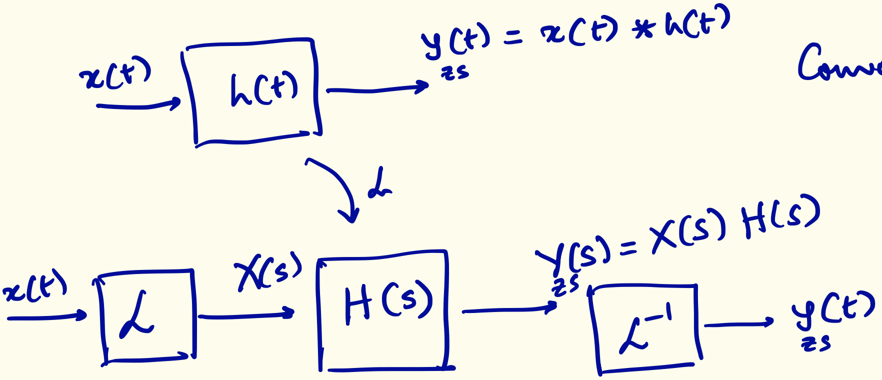
$t - \tau = t'$

$dt = dt'$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\, e^{-s\tau}\, d\tau \int_{-\infty}^{+\infty} x_2(t')\, e^{-st'}\, dt'$$

$$= \underset{X(s)}{X_1(s)} \quad \underset{H(s)}{X_2(s)}$$

Zero-state



$y_{zs}(t) = x(t) * h(t)$

Convolution

$Y_{zs}(s) = X(s)\, H(s)$

$$H(s) = \int_{0^-}^{\infty} h(t)\, e^{-st}\, dt \rightarrow \text{Transfer function}$$

$$\mathcal{L}(h(t))$$

Multiplication

Convolution

$$x_1(t) \cdot x_2(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \frac{1}{2\pi j} X_1(s) * X_2(s)$$

# Inverse Laplace Transform

- Rational Functions $\frac{P(s)}{Q(s)}$

$$\overset{N}{Q(D)}\, y(t) = \overset{M}{P(D)}\, x(t) \qquad M \le N$$

zero initial conditions

$$D^k y(t) \xrightarrow{\mathcal{L}} s^k Y(s)$$

$$D^k x(t) \xrightarrow{\mathcal{L}} s^k X(s)$$

$D = \frac{d}{dt}$

eg:

$$(D^2+3D+2)y(t) = Dx(t)$$

$$(s^2+3s+2)Y(s) = sX(s)$$

Zero-state Response

$$Q(s)\, Y_{zs}(s) = P(s)\, X(s)$$

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{P(s)}{Q(s)}$$

Transfer Function

$$h(t) = \mathcal{L}^{-1}\left(H(s)\right)$$

$$y_{zs}(t) = \mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}X(s)\right)$$

**eg. 1**

$$(D^2+3D+2)\,y(t) = D\,x(t) \qquad M<N$$

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{P(s)}{Q(s)} = \frac{s}{s^2+3D+2}$$

$$\frac{s}{(s+2)(s+1)} = \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

Partial Fractions

$k_1 = 2$ sub. $s=-2$ hiding $s+2$ on LHS

$k_2 = -1$ sub $s=-1$ hiding $s+1$ on LHS

$$e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}$$

$$H(s) = \frac{2}{s+2} - \frac{1}{s+1}$$

$$h(t) = \left(2e^{-2t} - e^{-t}\right)u(t)$$

**eg 2**

$$(D^2+3D+2)\,y(t) = (D^2+2D+2)\,x(t) \qquad M=N$$

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{P(s)}{Q(s)} = \frac{s^2+2s+2}{s^2+3s+2}$$

$$\frac{s^2+2s+2}{s^2+3s+2} = 1 + \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

$$H(s) = \frac{s^2+2s+2}{(s+2)(s+1)} = 1 + \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

$$k_1 = -2$$

$$k_2 = +1$$

$$H(s) = 1 - \frac{2}{s+2} + \frac{1}{s+1}$$

$$\mathcal{L}(\delta(t)) = \int_{0^-}^{\infty} \underbrace{\delta(t)\, e^{-st}}\, dt = e^{-s(0)} = 1$$

$$h(t) = \delta(t) - \left(2e^{-2t} + e^{-t}\right) u(t)$$

$M > N$

$$H(s) = \frac{s^2+2s+2}{s+2} = s + \frac{2}{s+2} = \left(s + \frac{2}{s+2}\right) X(s)$$

$Y_{zs}(s)$

$$h(t) = \left(\frac{d}{dt} + 2e^{-2t}\right) u(t)$$

$y_{zs}(t)$

$$= \left(\underbrace{\frac{d}{dt} x(t)} + \underbrace{2e^{-t} * x(t)}\right)$$

$$H(s) = \frac{P(s)}{Q(s)}$$

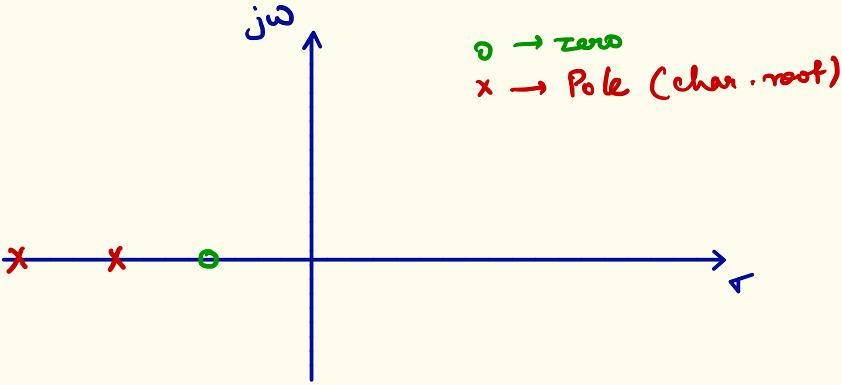
Roots of $P(s)$ → zeros of the system.

Roots of $Q(s)$ → Poles of the system

→ char. roots of the system.

eg.

$$H(s) = \frac{s+1}{(s+2)(s+3)}$$



Given $H(s) = \dfrac{P(s)}{Q(s)}$

Compute roots of $Q(s) = 0$ $\longrightarrow$ Poles
char.
roots

1. Asymptotically stable

   iff all poles lie on LHS of s-plane.
   (simple/repeated)

2. Marginally stable

   iff some simple poles on the $j\omega$-axis.
   Other poles are on LHS of s-plane.

3. Unstable

   iff a) some repeated poles on the $j\omega$-axis
   or/and
   b) some poles on RHS of s-plane.

<u>eg:</u> $H(s) = \dfrac{s}{s-2}$ Unstable

# Total Response

eg: $(D^2+3D+2)\,y(t) = (D+3)\,x(t)$

$y(0^-)=2, \quad \dot{y}(0^-)=1, \quad x(t) = e^{-4t}u(t)$

$X(s) = \frac{1}{s+4}$

Compute $y(t)$.

LT $\quad s^2Y(s) - \dot{y}(0^-) \underline{- s y(0^-)} + 3sY(s) - 3y(0^-) + 2Y(s)$

$= \frac{(s+3)}{s+4}$

$$(s^2+3s+2)\,Y(s) - 2s - 6 - 1 = \frac{s+3}{s+4}$$

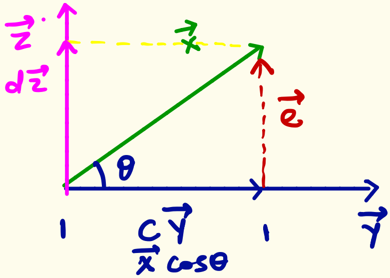
$$(s^2+3s+2)\,Y(s) = 2s+7 + \frac{s+3}{s+4}$$

$$Y(s) = \frac{2s+7}{s^2+3s+2} + \frac{s+3}{(s+2)(s+1)(s+4)}$$

$$Y(s) = Y_0(s) + Y_{zs}(s)$$

# Signal Representation

## Vectors



$\mathbb{R}^2$

$\vec{x} = c\vec{y} + \vec{e}$ ✓

$\vec{x} = c\vec{y} + d\vec{z}$

## Inner Product / Dot Product

$$\vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}| \cos\theta \quad ①$$

$\vec{y} = \vec{x}$

$$\vec{y} \cdot \vec{y} = |\vec{y}|^2 \cos\theta$$

$\theta = 0$

$$\vec{y} \cdot \vec{y} = |\vec{y}|^2 \quad ②$$

$$c|\vec{y}| = |\vec{x}| \cos\theta$$

$$c|\vec{y}|^2 = |\vec{x}|\,|\vec{y}| \cos\theta$$

From ① + ②

$$c\, \vec{y} \cdot \vec{y} = \vec{x} \cdot \vec{y} \Rightarrow c = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}} = \frac{\vec{x} \cdot \vec{y}}{|\vec{y}|^2}$$

$x(t), y(t) \longrightarrow$ signals $\quad t_1 \le t \le t_2$

$x(t) = c\, y(t) + e(t)$ for which $e(t)$ is min.

$e(t) = x(t) - c\, y(t)$

Find $c$ for which

$$\int_{t_1}^{t_2} e^2(t)\, dt \text{ is min.}$$

$$= \int_{t_1}^{t_2} (x(t) - c\, y(t))^2\, dt \longrightarrow \frac{\partial}{\partial c}(\,) = 0$$

$$\int_{t_1}^{t_2} \frac{\partial}{\partial c}\left[x^2(t) + c^2 y^2(t) - 2c\, x(t)\, y(t)\right] dt = 0$$

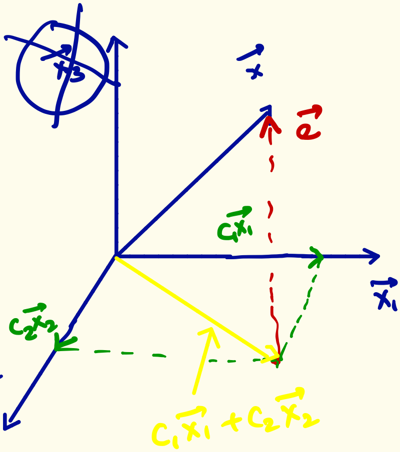
$$c^* = \frac{\int_{t_1}^{t_2} x(t)\, y(t)\, dt}{\int_{t_1}^{t_2} y^2(t)\, dt} = \frac{\int_{t_1}^{t_2} x(t)\, y(t)\, dt}{E_y}$$

$x_1(t), \quad x_2(t) \rightarrow$ orthogonal

$$\int_{t_1}^{t_2} x_1(t) \; x_2(t) \, dt \; = 0$$

$$\int_{t_1}^{t_2} \left(x_1(t) + x_2(t)\right)^2 dt \; = \int_{t_1}^{t_2} x_1^2(t) \, dt + \int_{t_1}^{t_2} x_2^2(t) \, dt$$

$E_{x_1+x_2} \; = \; E_{x_1} + E_{x_2}$ when $x_1, x_2$
→ orthogonal



$\vec{x} \in \mathbb{R}^3$

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{e}$$

$$c_1 = \frac{\vec{x} \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}$$

$$c_2 = \frac{\vec{x} \cdot \vec{x}_2}{\vec{x}_2 \cdot \vec{x}_2}$$

Include $\vec{x}_3$

$$c_3 = \frac{\vec{x} \cdot \vec{x}_3}{\vec{x}_3 \cdot \vec{x}_3} \qquad \vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

Signal $x(t) = \sum_{n=1}^{N} c_n x_n(t) + e(t)$

$$\int_{t_1}^{t_2} x_n(t)\, x_m(t)\, dt = \begin{cases} 0, & n \neq m \\ E_n, & n = m \end{cases}$$

$$e(t) = x(t) - \sum_{n=1}^{N} c_n x_n(t)$$

Find $c_1, c_2, \dots, c_N$ such that

$E_e = \int_{t_1}^{t_2} e^2(t)\, dt$ is minimum.

$= \int_{t_1}^{t_2} \left( x(t) - \sum_{n=1}^{N} c_n x_n(t) \right)^2 dt$ is minimum

$\rightarrow \dfrac{\partial (\ )}{\partial c_i} = 0 \quad \forall\ i = 1, 2, \dots, N$

$$c_n = \frac{\int_{t_1}^{t_2} x(t)\, x_n(t)\, dt}{E_n} \qquad E_n = \int_{t_1}^{t_2} x_n^2(t)\, dt$$

Sub

$$C_n E_n = \int_{t_1}^{t_2} x(t) x_n(t)\, dt \quad \text{in} \longrightarrow ①$$

$$\int_{t_1}^{t_2} \left( x(t) - \sum_{n=1}^{N} C_n x_n(t) \right)^2 dt$$

$$= \int_{t_1}^{t_2} \left[ x^2(t) + \left( \sum_{n=1}^{N} C_n x_n(t) \right)^2 - 2\, x(t) \sum_{n=1}^{N} C_n x_n(t) \right] dt$$

$$= \int_{t_1}^{t_2} x^2(t)\, dt + \int_{t_1}^{t_2} \overbrace{\left( \sum_{n=1}^{N} C_n x_n(t) \right)^2}^{N \times N} dt - 2C_n \sum_{n=1}^{N} \underbrace{\int_{t_1}^{t_2} x(t)\, x_n(t)\, dt}_{C_n E_n} \quad \text{From } ①$$

$N$

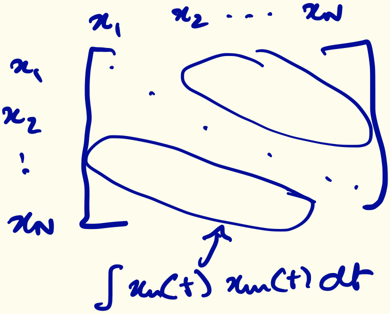
$$= \int_{t_1}^{t_2} x^2(t)\, dt + \int_{t_1}^{t_2} \sum_{n=1}^{N} C_n^2\, x_n^2(t)\, dt - 2 \sum_{n=1}^{N} C_n^2 E_n$$

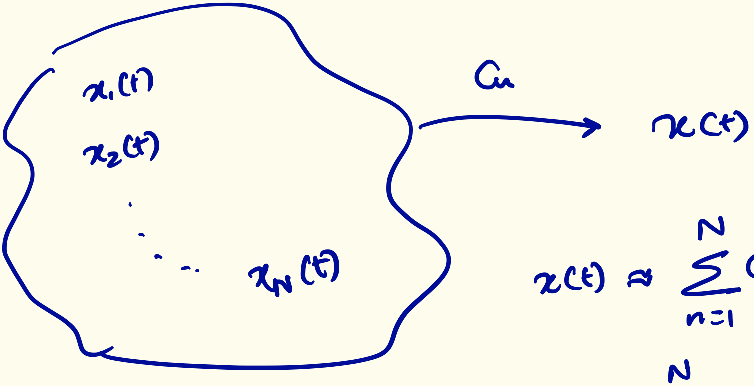
$$= E_x + \sum_{n=1}^{N} C_n^2 \underbrace{\int_{t_1}^{t_2} x_n^2(t)\, dt}_{E_n} - 2 \sum_{n=1}^{N} C_n^2 E_n$$

$$= E_x + \sum_{n=1}^{N} C_n^2 E_n - 2 \sum_{n=1}^{N} C_n^2 E_n$$

$$= E_x - \sum_{n=1}^{N} C_n^2 E_n$$

$$= \int_{t_1}^{t_2} e^2(t)\, dt \rightarrow \text{Energy of } e(t)$$





$$x(t) \approx \sum_{n=1}^{N} c_n \, x_n(t)$$

$$x(t) = \sum_{n=1}^{N} c_n x_n(t) + e(t)$$

As $N \to \infty$

$$E_e = 0$$

or

$$E_x = \sum_{n=1}^{\infty} c_n^2 E_n$$

$$\Longrightarrow \quad x(t) = \sum_{n=1}^{\infty} c_n \, x_n(t)$$

$$\int_{t_1}^{t_2} x_n(t)\, x_m(t)\, dt = \begin{cases} 0, & m \neq n \\ E_n, & m = n \end{cases}$$

# Trigonometric Fourier Series

$x(t) \qquad 0 \le t \le T_0$

$x_n(t)$

$$\Big\{ \overset{\cos 0t}{1}, \ \cos \omega_0 t, \ \cos 2\omega_0 t, \ \ldots$$

$$\sin \omega_0 t, \ \sin 2\omega_0 t, \ \ldots \Big\}$$

$\omega_0 = \dfrac{2\pi}{T_0}$ → radian frequency (fundamental)

$T_0$ → Period

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right)$$

$$\omega_0 T_0 = 2\pi$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{\infty} \Big( a_n \cos(n\omega_0 t + n\omega_0 T_0) + b_n \sin(n\omega_0 t + n\omega_0 T_0) \Big)$$

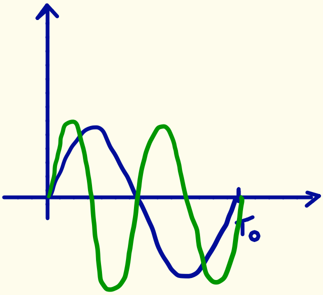
$$= a_0 + \sum_{n=1}^{\infty} \Big( a_n \cos(n\omega_0 t + 2\pi n) + b_n \sin(n\omega_0 t + 2\pi n) \Big)$$

$$= a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right) = x(t)$$

$x(t) \rightarrow$ Periodic with period $T_0$

$$\int_{T_0} x(t)\,dt = \int_{T_0} a_0\,dt + 0$$

$$= a_0 T_0$$



$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{\displaystyle\int_{T_0} x(t)\cos n\omega_0 t\,dt}{\displaystyle\int_{T_0} \cos^2 n\omega_0 t\,dt}$$

$$= \frac{\displaystyle\int_{T_0} x(t)\cos n\omega_0 t\,dt}{\displaystyle\int_{T_0}\left[\frac{1}{2}+\frac{\cos 2n\omega_0 t}{2}\right]dt} = \frac{\displaystyle\int_{T_0} x(t)\cos n\omega_0 t\,dt}{\displaystyle\frac{1}{2}\int_{T_0} dt}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t \, dt$$

## Orthogonality

$$\int_{T_0} \cos n\omega_0 t \sin m \omega_0 t \, dt$$

$$= \frac{1}{2} \int_{T_0} \left[ \sin (n+m)\omega_0 t + \sin (m-n) \omega_0 t \right] dt$$

$$= 0$$

$$\int_{T_0} \cos n \omega_0 t \cos m\omega_0 t \, dt$$

$$= \frac{1}{2} \int_{T_0} \left[ \cos (n+m) \omega_0 t + \cos (n-m) \omega_0 t \right] dt$$

$n \neq m$

$$= 0$$

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt$$

$$= \frac{1}{2} \int_{T_0} \left[ \cos (n-m) \omega_0 t - \cos (n+m) \omega_0 t \right] dt$$

$n \neq m$

$$= 0$$

$n=m$

$$\int_{T_0} \overset{\nearrow \frac{1}{2}+\frac{1}{2}\cos 2n\omega_0 t}{\cos^2 n\omega_0 t}\, dt = \frac{T_0}{2}$$

$$\int_{T_0} \sin^2 n\omega_0 t \, dt = \frac{T_0}{2}$$

$$\int_a^{a+T_0} dt = T_0 \quad , \quad \int_b^{b+T_0} dt = T_0$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t \, dt$$

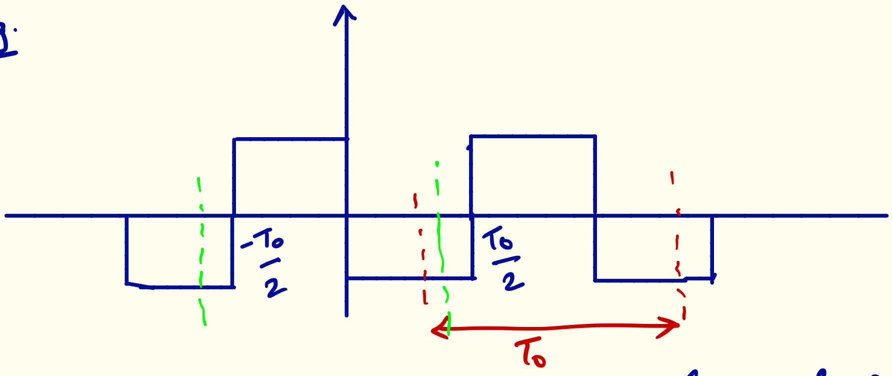
$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t \, dt$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

$x(t) \rightarrow$ even , $b_n = 0 \quad \forall\, n = 1, 2, \ldots$

$x(t) \rightarrow$ odd , $a_0 = 0$ , $a_n = 0 \quad \forall\, n = 1, 2, \ldots$

eg:



## Trigonometric Fourier Series – Compact form

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$= C_0 + \sum_{n=1}^{\infty} \left[ C_n \cos\theta_n \cos(n\omega_0 t) - C_n \sin\theta_n \sin n\omega_0 t \right]$$

$a_0 = C_0$

$a_n = C_n \cos\theta_n$

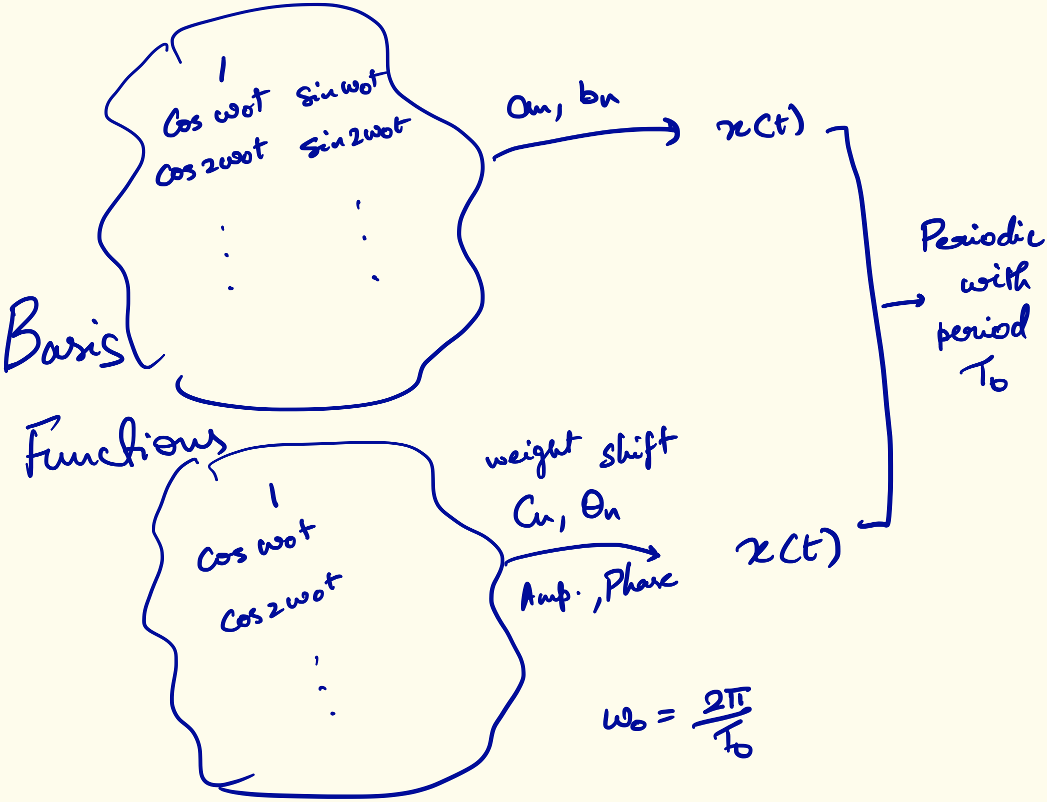
$b_n = -C_n \sin\theta_n$

$a_n^2 + b_n^2 = C_n^2$

$C_n = \sqrt{a_n^2 + b_n^2}$

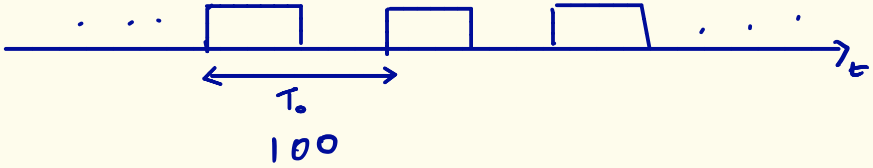
$-\frac{b_n}{a_n} = \tan\theta_n$

$\theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$

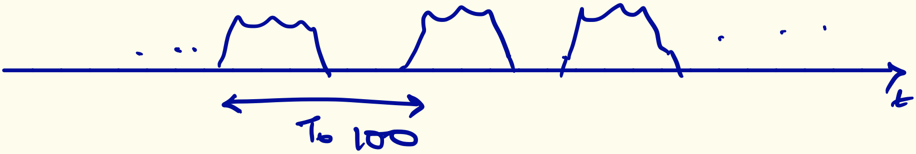


$$\omega_0 = \frac{2\pi}{T_0}$$

$x_1(t)$



$x_2(t)$



Periodic Signal with period $T_0$

$f_0 = \frac{1}{T_0}$ Hz $\quad \omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$ rad/s

Time Domain

$x(t) = x(t + kT_0)$

$k = 0, \pm 1, \pm 2, \ldots$

Frequency Domain

Basis Signals

$\{1, \cos\omega_0 t, \cos 2\omega_0 t, \ldots\}$

$n \rightarrow 0, \quad 1, \quad 2, \quad$

Amplitude spectrum

$\{C_0, C_1, C_2, \ldots\}$

Phase Spectrum

$\{0, \theta_1, \theta_2, \ldots\}$

$\omega_0 = \frac{2\pi}{T_0}$

Amplitude Spectrum

$n\omega_0$ Vs $C_n$

or $n$ Vs $C_n$

Phase Spectrum

$n\omega_0$ Vs $\theta_n$

$n$ Vs $\theta_n$

# Exponential Fourier Series

$x(t) \rightarrow$ Period $T_0$

$x_u(t)$

$$\left\{ e^{0t},\ e^{\pm j\omega_0 t},\ e^{\pm 2j\omega_0 t},\ \ldots \right\} \rightarrow \text{Basis Signals}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

$\vec{x}, \vec{y} \in \mathbb{C}^n$

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}^*$

$x(t), y(t) \rightarrow$ Complex

$\int_{T_0} x(t)\, y^*(t)\, dt$

$$D_n = \frac{\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt}{\int_{T_0} \underbrace{e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t}}_{1}\, dt}$$

$$D_n = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$\int_{T_0} e^{jn\omega_0 t}\, e^{-jm\omega_0 t}\, dt = \begin{cases} T_0, & n = m \\ 0, & n \neq m \end{cases}$$

$$\int_0^{T_0} e^{j(n-m)\omega_0 t}\, dt = \left. \frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0} \right|_0^{T_0} = 0$$

$$x(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} D_n (\cos n\omega_0 t + j \sin n\omega_0 t)$$

$$= D_0 + \sum_{n=1}^{\infty} D_n (\cos n\omega_0 t + j \sin n\omega_0 t)$$

$$+ \sum_{n=1}^{\infty} D_{-n} (\cos n\omega_0 t - j \sin n\omega_0 t)$$

$$D_0 = a_0$$

$$a_n = D_n + D_{-n}$$

$$b_n = j(D_n - D_{-n})$$

$$b_n = j(D_n - a_n + D_n)$$

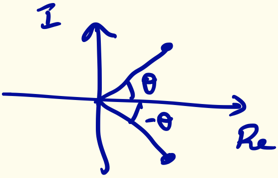
$$2j D_n = b_n + j a_n \qquad D_n = \frac{1}{2j}(-j^2 b_n + j a_n)$$

$$D_n = \frac{1}{2}(a_n - j b_n)$$

$$D_{-n} = a_n - D_n$$

$$= a_n - \frac{1}{2}(a_n - jb_n)$$

$$D_{-n} = \frac{1}{2}(a_n + jb_n)$$



$$D_0 = a_0 \quad , \quad D_n = \frac{1}{2}(a_n - jb_n) \quad , \quad D_{-n} = \frac{1}{2}(a_n + jb_n)$$
$$= C_0$$

$$|D_n| = |D_{-n}| = \sqrt{\frac{1}{4}(a_n^2 + b_n^2)} = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$= \frac{1}{2} C_n$$

$$C_n = \sqrt{a_n^2 + b_n^2} \quad , \quad \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

$$\angle D_n = -\angle D_{-n} = \theta_n$$

Amplitude spectrum

$|D_n|$ Vs $n\omega_0$ or $n$ Even

Phase spectrum

$\angle D_n$ Vs $n\omega_0$ or $n$ Odd

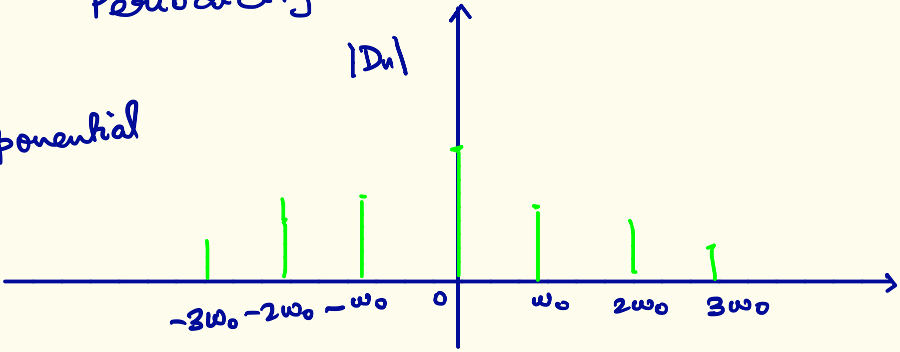
## Time Domain

## Frequency Domain

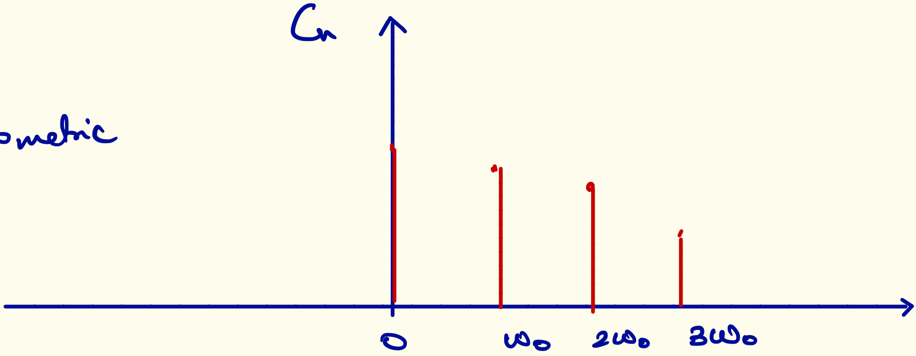
Periodicity

Discreteness

Exponential



Trigonometric



$$BW = 3\omega_0$$

## Parseval's Theorem

$$P_x = C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2$$

$$P_x = \sum_{n=-\infty}^{+\infty} |D_n|^2$$

Fourier Series
Conserver
Power

$$x_1(t) = C_1 \cos(\omega_0 t + \theta_1)$$

$$P_{x_1} = \lim_{T\to\infty} \frac{1}{T} \int_0^T C_1^2 \cos^2(\omega_0 t + \theta_1)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T C_1^2 \left[\frac{1}{2} + \underset{\substack{\downarrow \\ 0}}{\frac{\cos 2(\omega_0 t + \theta_1)}{2}}\right] dt$$

$$= \frac{1}{\cancel{T}} \frac{C_1^2 \cancel{T}}{2}$$

$$= \frac{C_1^2}{2}$$

$$x_1(t) + x_2(t) = C_1 \cos(\omega_0 t + \theta_1) + C_2 \cos(2\omega_0 t + \theta_2)$$

$$P_{x_1+x_2} = \frac{C_1^2}{2} + \frac{C_2^2}{2}$$

$$x(t) = D_1 e^{j\omega_0 t}$$

$$P_{x_1} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left|D_1 e^{j\omega_0 t}\right|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T |D_1|^2\, dt$$

$$= \frac{1}{\cancel{T}} |D_1|^2 \cancel{T} = |D_1|^2$$

$$x_1(t) + x_2(t) = D_1 e^{j\omega_0 t} + D_2 e^{j2\omega_0 t}$$

$$P_{x_1+x_2} = |D_1|^2 + |D_2|^2$$

## Convergence in Mean

$x(t) \longrightarrow$ Periodic with $T_0$

$$\sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

$$x_N(t) = \sum_{n=-N}^{N} D_n e^{jn\omega_0 t}$$

$$\int_{T_0} |x(t) - x_N(t)|^2 dt \longrightarrow 0 \quad \text{as} \quad N \to \infty$$

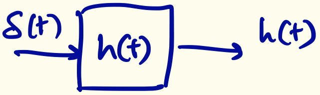
$$D_n < \infty$$

$$D_n = \frac{1}{T_0} \int_{T_0} x(t) \underline{e^{-jn\omega_0 t}} dt$$

$$\cos n\omega_0 t - j \sin n\omega_0 t$$

BIBO
Stable

$$\int_{T_0} |x(t)| dt < \infty$$



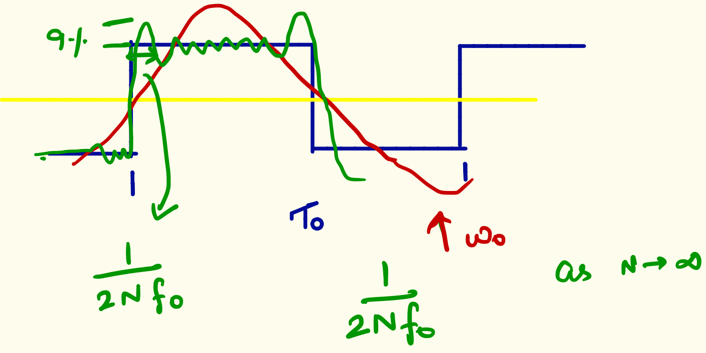
# Dirichlet Conditions for Periodic $x(t)$ to have a FS

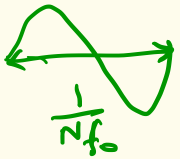
1. $x(t)$ is absolutely integrable over period $T_0$. $\int_{T_0} |x(t)|dt < \infty$

2. $x(t)$ has finite number of finite discontinuities over period $T_0$.

3. $x(t)$ has finite number of maxima and minima over period $T_0$.

## Gibbs Phenomenon



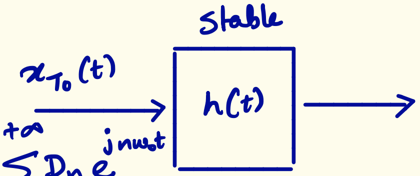


$Nf_0 \to$ freq.

Periodic Signal $\longrightarrow x_{T_0}(t)$

Aperiodic Signal $\longrightarrow x(t)$

LTI System



$$= \sum_{n=-\infty}^{+\infty} D_n \underline{e}^{jn\omega_0 t}$$

zero state

$H(s)$

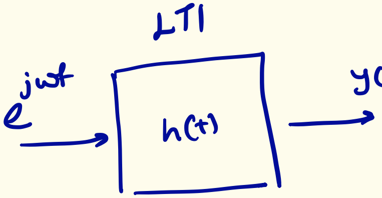
$\cancel{\sigma} + j\omega = s$

$H(j\underline{\omega})$

$$y_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \underbrace{D_n H(jn\omega_0)}_{D_n'} e^{jn\omega_0 t}$$

$$y_{T_0}(t) = \sum_{n=-\infty}^{+\infty} D_n' e^{jn\omega_0 t}$$

$$D_n' = D_n H(jn\omega_0)$$



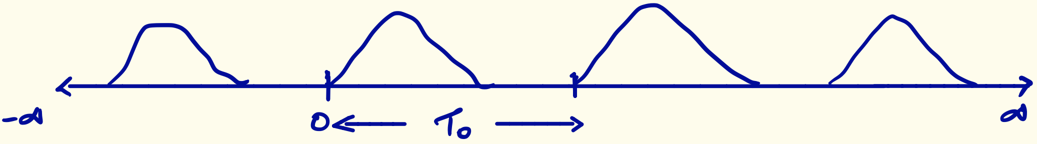
$$y(t) = \int_{-\infty}^{+\infty} h(\tau) e^{j\omega(t-\tau)} d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

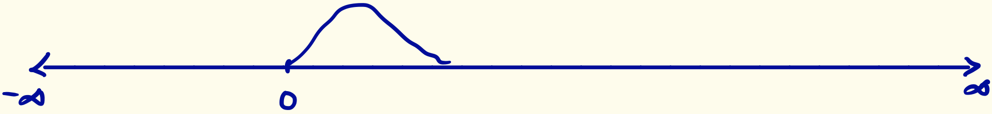
$$= e^{j\omega t} H(j\omega)$$

# Fourier Transform (CT)

Periodic $x_{T_0}(t)$



Aperiodic $x(t)$

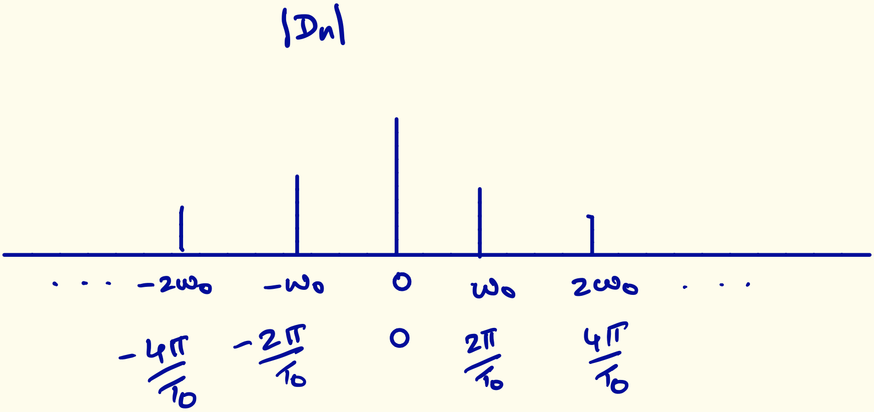


$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} D_n \, e^{jn\omega_0 t} \quad \text{———} \; ①$$

Define $$X(\omega) = \int_{-\infty}^{+\infty} x(t) \, e^{-j\omega t} \, \underset{\hookrightarrow \text{FT}}{dt} \quad \text{———} \; ②$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) \, e^{-jn\omega_0 t} \, dt \quad \text{———} \; ③$$



As $T_0 \to \infty$,

1. Spacing between $\underset{\Delta\omega}{\text{samples}} \longrightarrow 0$

2. Amplitude of spectrum $\longrightarrow 0$

$\lim_{T_0\to\infty} D_n$

$$= \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t}\, dt = \lim_{T_0\to\infty} \frac{1}{T_0} \underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-jn\omega_0 t}\, dt}$$

$$= \lim_{T_0\to\infty} \frac{1}{T_0} X(n\omega_0) \longrightarrow ④$$

$x(t) =$

① $\Rightarrow \lim_{T_0\to\infty} x_{T_0}(t) = \lim_{T_0\to\infty} \sum_{n=-\infty}^{+\infty} D_n\, e^{jn\omega_0 t}$

From ④

$$x(t) = \lim_{T_0\to\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{T_0} X(n\omega_0)\, e^{jn\omega_0 t}$$

$$T_0 = \frac{2\pi}{\Delta\omega} \quad \text{as} \quad T_0 \to \infty$$

$$\Delta\omega \to 0$$

$$\Delta\omega = \omega_0$$

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} X(n\,\Delta\omega)\, e^{jn\Delta\omega t}\, \Delta\omega$$

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

IFT

as $\Delta\omega \to 0$

$n\Delta\omega \to \omega$

$\sum_n \to \int_\omega$

Given $x(t)$

CTFT $$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \rightarrow \text{complex}$$

ICTFT $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$x(t) \rightarrow$ real

$$|X(\omega)| = |X(-\omega)| \quad \text{even} \quad \text{Magnitude}$$

$$\angle X(\omega) = -\angle X(-\omega) \quad \text{odd} \quad \text{Phase}$$

**Eg.**

1. $x(t) = \delta(t)$

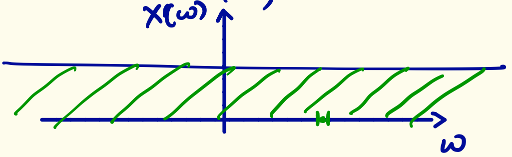
$$X(\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t)\, dt$$

at $t=0$, $e^{-j\omega(0)} = 1$

$t \neq 0$, $\delta(t) = 0$

$$= 1$$



2. $X(\omega) = \delta(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}$$

1. $X(\omega) \rightarrow$ Continuous in $\omega$ as $x(t)$ is aperiodic

2. $X(\omega) \rightarrow$ Spectral density $\left(\frac{1}{2\pi} X(n\Delta\omega)\Delta\omega\right) e^{jn\Delta\omega t}$

$x(t)$ $$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\hat{x}(t) = \lim_{W \to \infty} \frac{1}{2\pi} \int_{-W}^{+W} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\int_{t=-\infty}^{+\infty} \left| x(t) - \hat{x}(t) \right|^2 dt \longrightarrow 0$$

True iff $\int_{t=-\infty}^{+\infty} |x(t)|\, dt < \infty$ Absolutely Integrable.

**Dirichlet Conditions** for existence of CTFT of $x(t)$

**Sufficient**

1. Absolutely Integrable $\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$

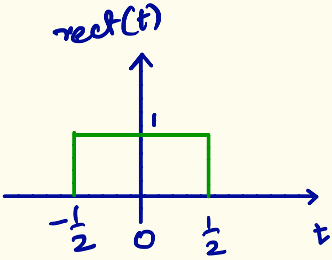
2. Any finite duration → finite number of finite discontinuities in $x(t)$

3. Any finite duration → finite number of maxima and minima.

# Basic Signals

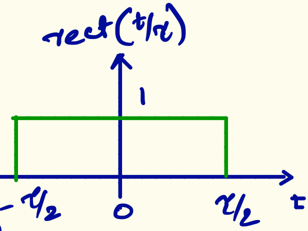
## 1. Rectangular

$$\mathrm{rect}(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & |t| = \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$



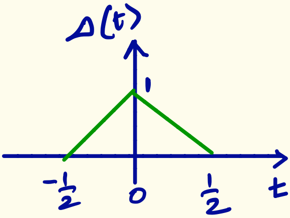
$$\mathrm{rect}(t/\tau) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ \frac{1}{2}, & |t| = \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

$\tau > 0$



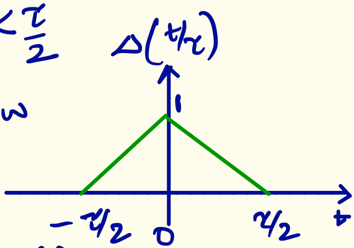
## 2. Triangular

$$\Delta(t) = \begin{cases} 1-2|t|, & |t| < \frac{1}{2} \\ 0, & \text{o/w} \end{cases}$$



$$\Delta(t/\tau) = \begin{cases} 1-2|t/\tau|, & |t| < \frac{\tau}{2} \\ 0, & \text{o/w} \end{cases}$$

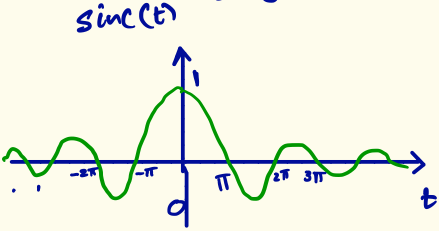
$\tau > 0$



## 3. Sinc

$$\mathrm{sinc}(t) = \frac{\sin t}{t}$$

1. At $t=0$, $\mathrm{sinc}(0) = 1$
2. $\mathrm{sinc}(t) = 0$, $t = \pm n\pi$, $n = 1, 2, \ldots$
3. $\mathrm{sinc}(t) = \sin t \cdot \frac{1}{t}$



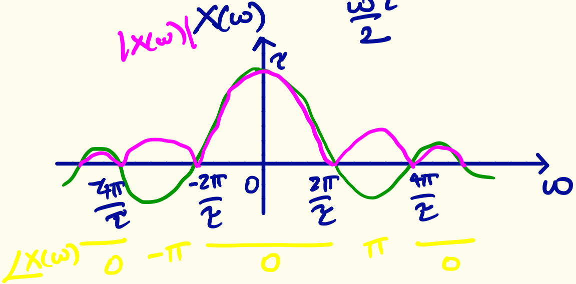
**eg. 3** $x(t) = \text{rect}(t/\tau)$

$$X(\omega) = \int_{-\infty}^{+\infty} \text{rect}(t/\tau)\, e^{-j\omega t}\, dt$$

$$= \int_{-\tau/2}^{+\tau/2} e^{-j\omega t}\, dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\tau/2}^{\tau/2}$$

$$= \frac{\cancel{2}}{2j\omega}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right] = \frac{\tau \sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$$

$$= \tau\, \text{sinc}\left(\frac{\omega\tau}{2}\right)$$



$$\frac{\omega\tau}{2} = n\pi \qquad \omega = \frac{2n\pi}{\tau}$$

**eg. 4** $X(\omega) = \delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$\delta(\omega-\omega_0) = \begin{cases} \text{und}; & \omega = \omega_0 \\ 0, & \text{o/w} \end{cases}$$

$$= \frac{e^{j\omega_0 t}}{2\pi}\int_{-\infty}^{+\infty} \delta(\omega-\omega_0)\, d\omega = \frac{e^{j\omega_0 t}}{2\pi}$$

$$2\pi\,\delta(\omega-\omega_0) \underset{\longleftarrow}{\xrightarrow{\mathcal{F}^{-1}}} e^{j\omega_0 t}$$

$$2\pi\,\delta(\omega+\omega_0) \underset{\xleftarrow[\mathcal{F}]{}}{\overset{\mathcal{F}}{\xrightarrow{\mathcal{F}^{-1}}}} e^{-j\omega_0 t}$$

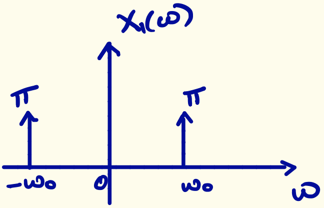
**eg.5** $x_1(t) = \cos\omega_0 t$ $x_2(t) = \sin\omega_0 t$

$$X_1(\omega) = \mathcal{F}[\cos\omega_0 t]$$

$$= \mathcal{F}\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{1}{2}\left[2\pi\,\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

$$= \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$



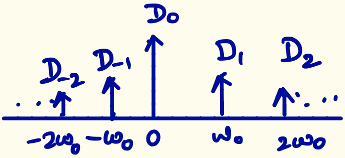
$$X_2(\omega) = \pi j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

**eg.6** Periodic $x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$ → Fourier Series

Fourier transform
↓

$$\mathcal{F}[x_{T_0}(t)] = \mathcal{F}\left[\sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{+\infty} D_n\, \mathcal{F}\left[e^{jn\omega_0 t}\right]$$



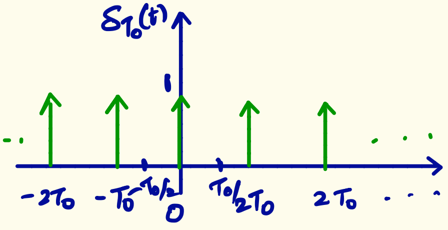
$$= 2\pi \sum_{n=-\infty}^{+\infty} D_n\, \underbrace{\delta(\omega - n\omega_0)}_{\neq 0 \text{ iff } \omega = n\omega_0}$$

**eg.7** Unit Impulse train $\delta_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT_0)$

FS

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$$

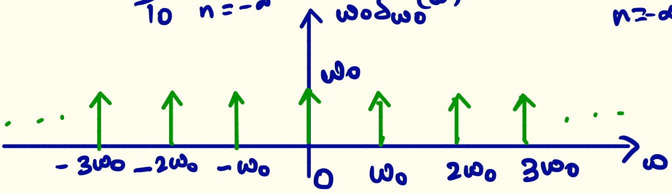
↳ $\neq 0$ at $t=0$



$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta_{T_0}(t)\, dt = \frac{1}{T_0}$$

$$\mathcal{F}[\delta_{T_0}(t)] = \frac{2\pi}{T_0}\sum_{n=-\infty}^{+\infty}\delta(\omega - n\omega_0) = \omega_0\sum_{n=-\infty}^{+\infty}\delta(\omega - n\omega_0)$$

$$= \omega_0\,\delta_{\omega_0}(\omega)$$



**eg.8** $x(t) = e^{-at}u(t) \quad a>0$

$$X(\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}\,dt = \int_0^{\infty} e^{-(a+j\omega)t}\,dt = \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty}$$

$$= \frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle X(\omega) = \tan^{-1}(-\omega/a)$$

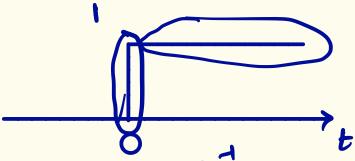
**eg.9** $x(t) = u(t)$ Unit Step

$$X(\omega) = \int_0^{\infty} e^{-j\omega t}\,dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^{\infty} = \frac{1}{j\omega}$$

Prob at $\omega=0$

$\times$ for $t\to\infty$

$$x(t) = \lim_{a\to 0} e^{-at}u(t) = u(t)$$



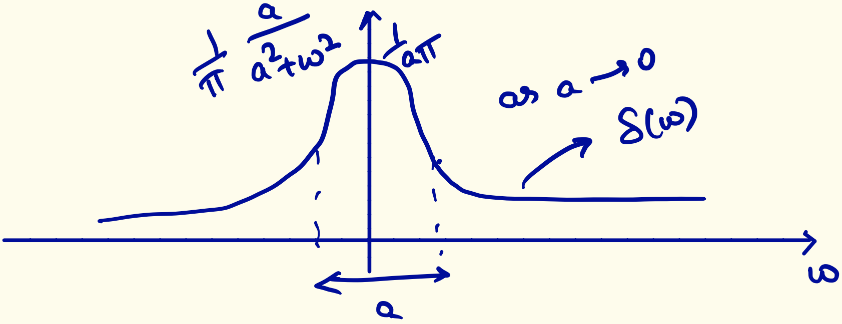
$$X(\omega) = \lim_{a\to 0}\frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2} \qquad \delta(\omega) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2\pi}$$

$$= \lim_{a\to 0}\left(\frac{a}{a^2+\omega^2}\right) + \frac{1}{j\omega} = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\int_{-\infty}^{+\infty}\frac{a}{a^2+\omega^2}\,d\omega = \left.\tan^{-1}\left(\frac{\omega}{a}\right)\right|_{-\infty}^{+\infty} = \pi$$

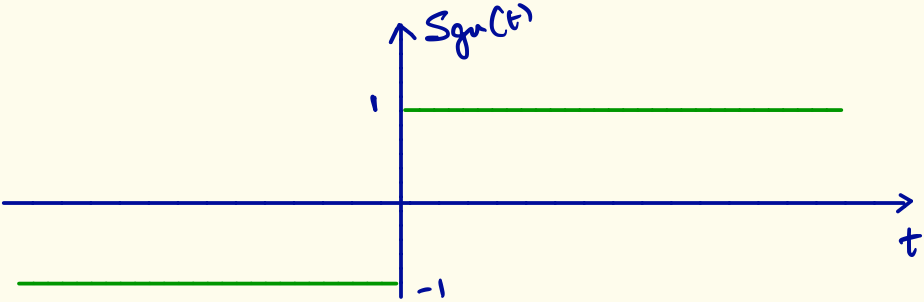
area at $\omega=0$

$$\delta(t) = \frac{1}{\pi} \lim_{a \to 0} \frac{a}{a^2+t^2}$$



**eg. 10**

$$\mathrm{Sgn}(t) = 2u(t) - 1$$



$$\mathcal{F}[\mathrm{Sgn}(t)] = 2\mathcal{F}[u(t)] - 2\pi\delta(\omega)$$

$$= 2\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - 2\pi\delta(\omega)$$

$$= \frac{2}{j\omega}$$

$$\frac{1}{jw} \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}\mathrm{Sgn}(t)$$

# Properties of CTFT

## 1. Linearity

$$x_1(t) \xrightarrow{\mathcal{F}} X_1(\omega)$$

$$x_2(t) \xrightarrow{\mathcal{F}} X_2(\omega)$$

$$a_1 x_1(t) + a_2 x_2(t) \xrightarrow{\mathcal{F}} a_1 X_1(\omega) + a_2 X_2(\omega)$$

## 2. Duality

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(t) \xrightarrow{\mathcal{F}} 2\pi\, x(-\omega)$$

**Proof**

$$\underline{\mathcal{F}\left[X(t)\right]} = \int_{-\infty}^{+\infty} X(t)\, e^{-j\omega t}\, dt \quad \text{———} ①$$

FT

$$X(\omega) = \mathcal{F}\left[x(t)\right] = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

IFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad \text{———} ②$$

in ② $t \rightarrow \omega$ $\omega \rightarrow t$

$$\underline{2\pi\, x(-\omega)} = \int_{-\infty}^{+\infty} X(t)\, e^{-j\omega t}\, dt \quad \text{———} ③$$

**eg:**

$$\text{rect}\left(\frac{t}{\tau}\right) \xrightarrow{\mathcal{F}} \tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\tau \,\text{sinc}\left(\frac{t\tau}{2}\right) \xrightarrow{\mathcal{F}} 2\pi \,\text{rect}\left(\frac{-\omega}{\tau}\right)$$

$$= 2\pi \,\text{rect}\left(\frac{\omega}{\tau}\right)$$

**eg:** $x(t) = e^{at} u(-t)$ , $a>0$

## 3. Time scaling

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{+\infty} x(at)\, e^{-j\omega t} dt$$

$t' = at$

$\frac{dt'}{a} = dt$

$a>0$

$$a>0 \qquad = \frac{1}{a}\int_{-\infty}^{+\infty} x(t')\, e^{-j\frac{\omega}{a}t'} dt'$$

$t = -\infty$ , $t' = -\infty$

$t = \infty$ , $t' = \infty$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

$$a<0 \qquad \frac{1}{a}\int_{\infty}^{-\infty} x(t')\, e^{-j\frac{\omega}{a}t'} dt'$$

$t = \infty$ , $t' = -\infty$

$t = -\infty$ , $t' = \infty$

$$= \frac{1}{-a} X\left(\frac{\omega}{a}\right)$$

$$-1 \qquad \mathcal{F}[x(at)]$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$|a| > 1 \longrightarrow$ Compression

$|a| < 1 \longrightarrow$ Expansion

eg:

$$\cos \omega_0 t \xrightarrow{\mathcal{F}} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$\cos 2\omega_0 t \xrightarrow{\mathcal{F}} \pi\left[\delta(\omega-2\omega_0) + \delta(\omega+2\omega_0)\right]$$

$$= \frac{1}{2}\left[e^{j2\omega_0 t} + e^{-j2\omega_0 t}\right]$$

$$\cos \frac{2}{\cancel{2}}\omega_0 t \xrightarrow{\mathcal{F}} \frac{1}{2}\pi\left[\delta\left(\frac{\omega}{2}-\omega_0\right) + \delta\left(\frac{\omega}{2}+\omega_0\right)\right]$$

$$= \frac{\pi}{2}\left[\delta\left(\frac{\omega-2\omega_0}{2}\right) + \delta\left(\frac{\omega+2\omega_0}{2}\right)\right]$$

$$\delta(at) = \frac{1}{|a|}\delta(t) \qquad = \frac{\frac{\pi}{\cancel{2}}}{\frac{1}{\cancel{2}}}\left[\delta(\omega-2\omega_0) + \delta(\omega+2\omega_0)\right]$$

4. $x(t) \xrightarrow{\mathcal{F}} X(\omega)$

$$x(-t) \xrightarrow{\mathcal{F}} X(-\omega)$$

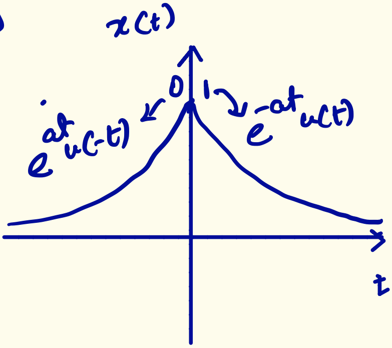
eg: $x(t) = e^{at}\,u(-t)$, $a > 0$

$$X(\omega) = \frac{1}{a-j\omega}$$

**eg.**

$$x(t) = e^{-at}u(t) + e^{at}u(-t)$$

$$a>0 \qquad = e^{-a|t|}$$



$$X(\omega) = \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

5. Time shifting

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$x(t-t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$$

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j\omega t}\, dt \qquad t' = t-t_0$$

$$dt' = dt$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j\omega(t'+t_0)}\, dt'$$

$$= e^{-j\omega t_0} \int_{-\infty}^{+\infty} x(t')\, e^{-j\omega t'}\, dt'$$

$$= e^{-j\omega t_0}\, X(\omega)$$

$$= (\cos\omega t_0 - j\sin\omega t_0)\, X(\omega)$$

- Magnitude spectrum is same.
- Phase spectrum $\theta(\omega) = -\omega t_0 \leftarrow \tan^{-1}(\tan(\omega t_0))$

eg:

$$x(t) = \cos \omega_0 (t - t_0)$$

$$X(\omega) = e^{-j\omega t_0} \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\theta(\omega) = -\omega t_0 \longrightarrow \text{Linear Phase}$$

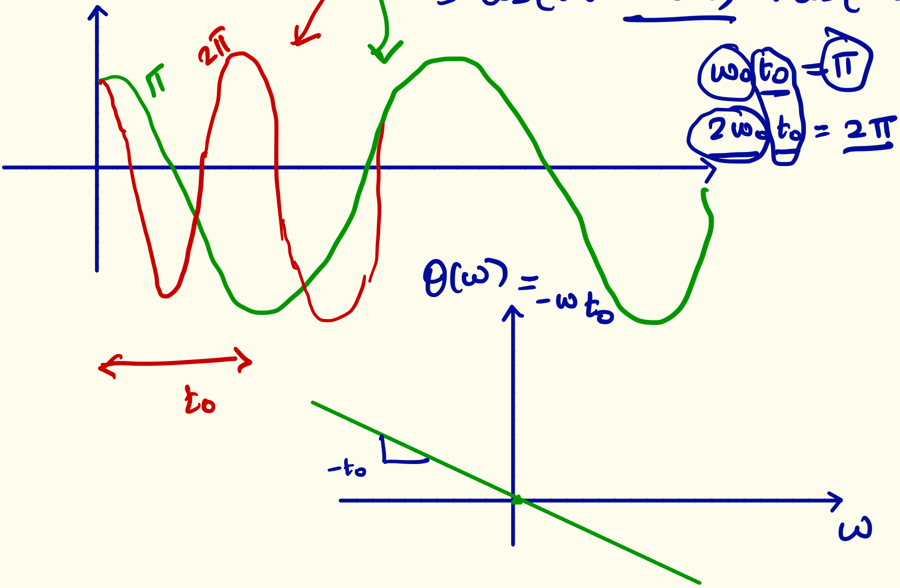
$$x_1(t) = \cos \omega_0 t + \cos 2\omega_0 t$$

$$x_2(t) = x_1(t - t_0) = \cos \omega_0 (t - t_0) + \cos 2\omega_0 (t - t_0)$$

$$= \cos(\omega_0 t - \underline{\omega_0 t_0}) + \cos(2\omega_0 t - \underline{2\omega_0 t_0})$$

$$\boxed{\omega_0} \boxed{t_0} = \boxed{\pi}$$

$$\boxed{2\omega_0} \boxed{t_0} = \underline{2\pi}$$



## 6. Frequency shifting

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$x(t)\, e^{j\omega_0 t} \xrightarrow{\mathcal{F}} X(\omega-\omega_0)$$

$$\mathcal{F}\left[x(t)\, e^{j\omega_0 t}\right] = \int_{-\infty}^{+\infty} x(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt$$

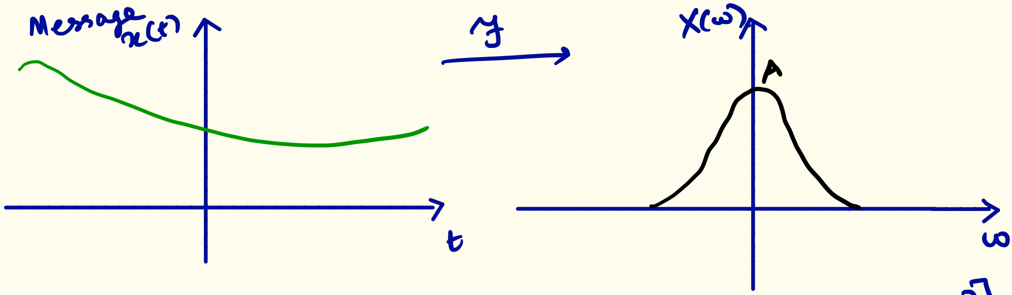
$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

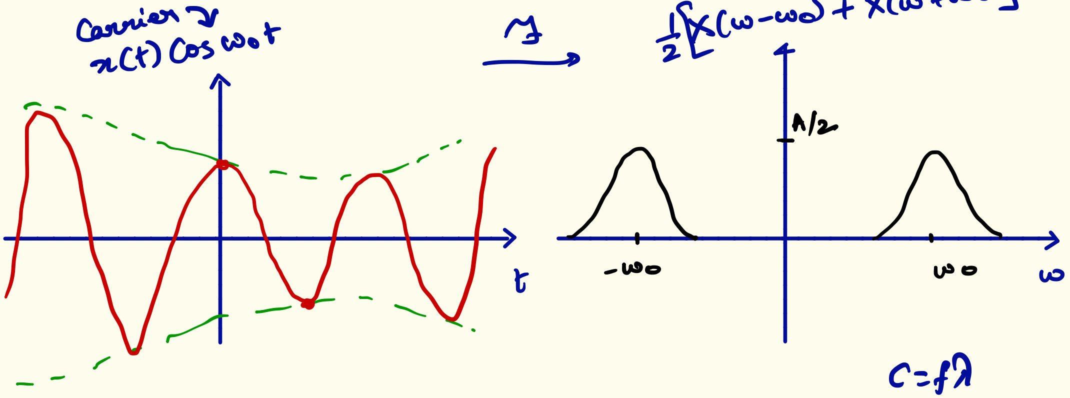
$$= X(\omega-\omega_0)$$

$$x(t)\, e^{-j\omega_0 t} \xrightarrow{\mathcal{F}} X(\omega+\omega_0)$$

$$x(t)\,\cos\omega_0 t \xrightarrow{\mathcal{F}} \frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$$

$$\frac{1}{2}\left[x(t)\, e^{j\omega_0 t} + x(t)\, e^{-j\omega_0 t}\right]$$





$c = f\lambda$

↑

$$x(t)\cos\omega_0 t = \begin{cases} x(t) & \cos\omega_0 t = +1 \\ -x(t) & \cos\omega_0 t = -1 \end{cases}$$

Amplitude Modulation → DSB - SC

- Frequency Division Multiplexing

- Antenna Design → small wavelength

**Demodulation**

$$x(t)\cos^2\omega_0 t = x(t)\left[\frac{1+\cos 2\omega_0 t}{2}\right] = \boxed{\frac{x(t)}{2}} + \cancel{\frac{x(t)\cos 2\omega_0 t}{2}}$$

## 7. Convolution

$$x_1(t) * x_2(t) \xrightarrow{\mathcal{F}} X_1(\omega)\, X_2(\omega)$$

$$x_1(t)\, x_2(t) \xleftarrow{\mathcal{F}^{-1}} \frac{1}{2\pi} X_1(\omega) * X_2(\omega) \qquad H$$

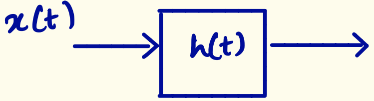
$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{+\infty} x_1(\tau) \left[ \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} X_2(\omega)\, e^{j\omega(t-\tau)}\, d\omega \right] d\tau$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} X_2(\omega)\, e^{j\omega t} \underbrace{\int_{\tau=-\infty}^{+\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau}_{X_1(\omega)}\, d\omega$$

$$x_1(t) * x_2(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} \underbrace{X_1(\omega)\, X_2(\omega)}_{ICTFT}\, e^{j\omega t}\, d\omega$$

LTI → BIBO stable
Marginally stable except resonance



$y_{zs}(t) = x(t) * h(t)$

$$Y(\omega) = X(\omega)\, H(\omega)$$

Frequency Response

$$\underline{H(\omega) = \mathcal{F}[h(t)]}$$

## 8. Integration

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$x(t) * \underbrace{u(t)}_{} = \int_{-\infty}^{t} x(\tau)\,d\tau \xrightarrow{\mathcal{F}} X(\omega)\overbrace{\left[\underline{\pi\delta(\omega)} + \frac{1}{j\omega}\right]}^{H(\omega)}$$

$$\begin{tikzcd} X(\omega) & \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] \\ x(t) \arrow[u, "\mathcal{F}"'] & u(t) \arrow[u, "\mathcal{F}"'] \end{tikzcd}$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{+\infty} x(\tau)\,u(t-\tau)\,d\tau = x(t) * u(t)$$

LPF

## 9. Differentiation

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$\frac{dx(t)}{dt} \xrightarrow{\mathcal{F}} \overbrace{j\omega}^{H(\omega)}\, X(\omega)$$

$$\text{assuming} \int_{-\infty}^{+\infty} \left|\frac{dx(t)}{dt}\right| dt < \infty$$

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = \int_{-\infty}^{+\infty} \frac{d\overset{v}{x(t)}}{dt}\, \overbrace{e^{-j\omega t}}^{u}\, dt = \underbrace{e^{-j\omega t}\, x(t)\Big|_{-\infty}^{+\infty}}_{0} - \int_{-\infty}^{+\infty} (-j\omega)\, x(t)\, e^{-j\omega t}\, dt$$

$$= j\omega \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = j\omega\, X(\omega)$$

HPF

$$\frac{d^n x(t)}{dt^n} \xrightarrow{\mathcal{F}} (j\omega)^n X(\omega)$$

# Distortionless Transmission

$Y(\omega) = H(\omega) X(\omega)$

$= k X(\omega) e^{-j\omega t_d}$

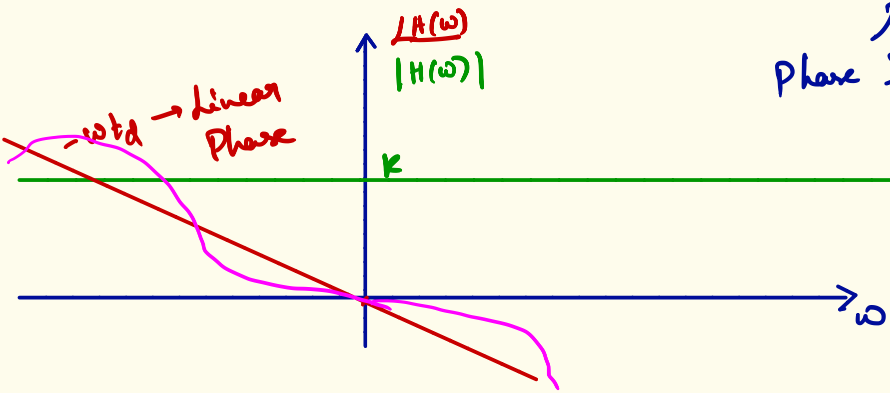
$y(t) = h(t) * x(t)$

$= k\, x(t - t_d)$

- Attenuation or Boosting of amplitude of $x(t)$.
- Time delay

$h(t) = k\, \delta(t - \overset{\uparrow}{t_d})$

Phase Delay



# Energy of $x(t)$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \, x^*(t) \, dt$$

$$= \int_{t=-\infty}^{+\infty} x(t) \, \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} X^*(\omega) \, e^{-j\omega t} \, d\omega \quad dt$$

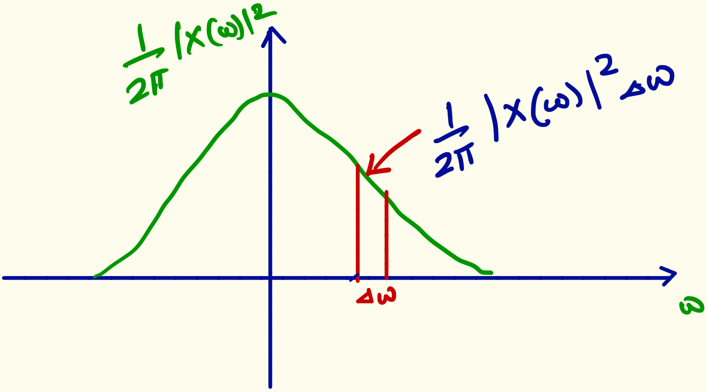
$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} X^*(\omega) \left[ \int_{t=-\infty}^{+\infty} x(t) \, e^{-j\omega t} \, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} X(\omega) \, X^*(\omega) \, d\omega$$

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega$$

$|X(\omega)|^2$ → Energy Spectral Density

Parseval's theorem



Time Unlimited $\overset{\text{mostly}}{\Longrightarrow}$ Band Limited

eg. $\cos\omega_0 t$,

$\sin\omega_0 t$,

$1$

$-\infty < t < \infty$

exceptions

- $\frac{1}{\sqrt{2\pi}} e^{-t^2/2\sigma^2} \rightarrow$ Gaussian
- Unit Impulse Train

$$\overset{x(t)}{\boxed{\cos\omega_0 t}} \cdot \text{rect}\left(t/\tau\right) \xrightarrow{\mathcal{F}} \frac{1}{2\pi}\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] * \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

Limit in Time

Unlimited in Frequency

$$\mathcal{F}\left[\frac{1}{2\pi} e^{-t^2/2\sigma^2}\right] \longrightarrow \text{HW}$$

Practical Signals

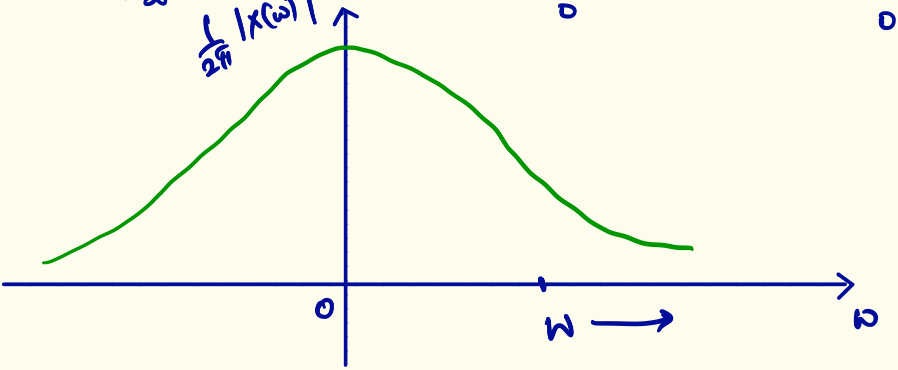
Time Limited $\Longrightarrow$ Band Unlimited

# Parseval's Theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega \rightarrow 100\% \text{ Energy}$$

For real $x(t)$, $|X(-\omega)| = |X(\omega)| \rightarrow$ even

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega = 2 \cdot \frac{1}{2\pi}\int_{0}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{\pi}\int_{0}^{\infty} |X(\omega)|^2 d\omega$$



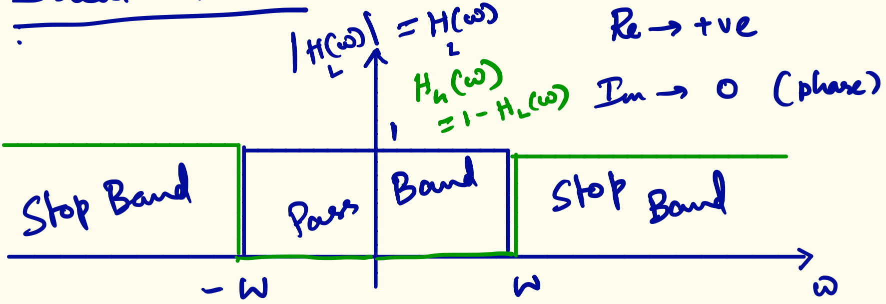
Compute $$\underbrace{\frac{1}{\pi}\int_{0}^{W} |X(\omega)|^2 d\omega}$$ while increasing W

as $W \rightarrow \infty$ → $E_x$

Find $W_e$ such that $$\frac{1}{\pi}\int_{0}^{W_e} |X(\omega)|^2 d\omega = 0.95\, E_x$$

$W_e$ → Essential Bandwidth of $x(t)$

# Ideal Filters



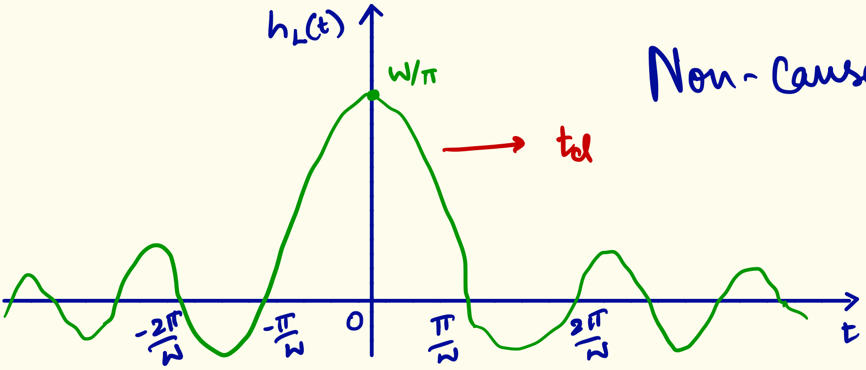
$$H_L(\omega) = \text{rect}\left(\frac{\omega}{2W}\right)$$

$\frac{\tau}{2} \rightarrow W$

$$h_L(t) = \mathcal{F}^{-1}\left(H_L(\omega)\right)$$

$$= \frac{1}{2\pi}\, 2W \,\text{sinc}\left(Wt\right)$$

$$h_L(t) = \frac{W}{\pi}\,\text{sinc}\left(Wt\right)$$



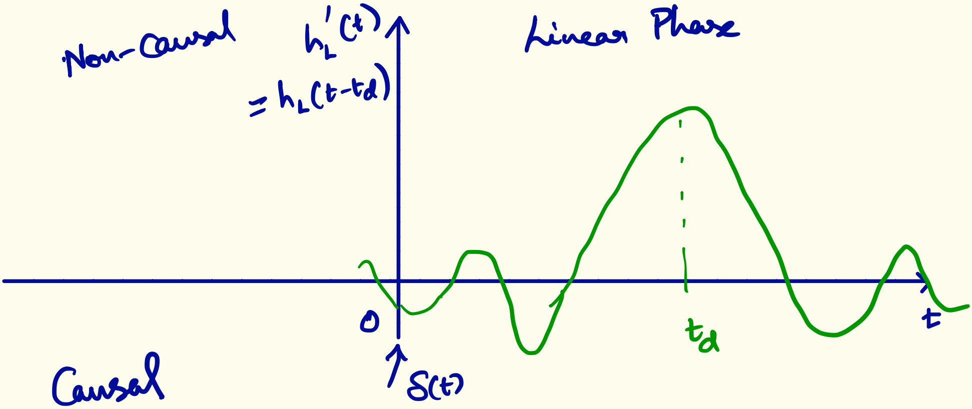
Non-Causal

With a delay $t_d$

$$h_L'(t) = \frac{W}{\pi} \; \text{sinc}\left(W(t-t_d)\right)$$

$$H_L'(\omega) = \text{rect}\left(\frac{\omega}{2W}\right) \underbrace{e^{-j\omega t_d}}$$

Linear Phase

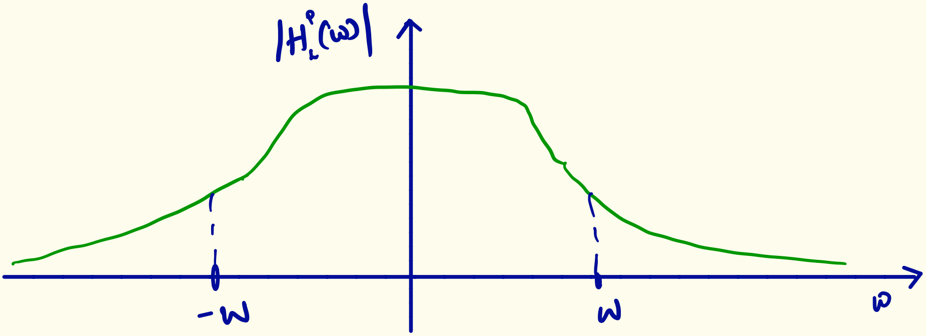


Causal

$$h_L^P(t) = h_L'(t)\, u(t) \xrightarrow{\mathcal{F}} H_L'(\omega) * \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\text{BW} \qquad \underbrace{2W \qquad\qquad \infty}_{H_L^P(\omega)}$$

$$\left.\begin{matrix} H_L^P(\omega) \\ h_L^P(t) \end{matrix}\right\} \longrightarrow \text{Practical LPF}$$



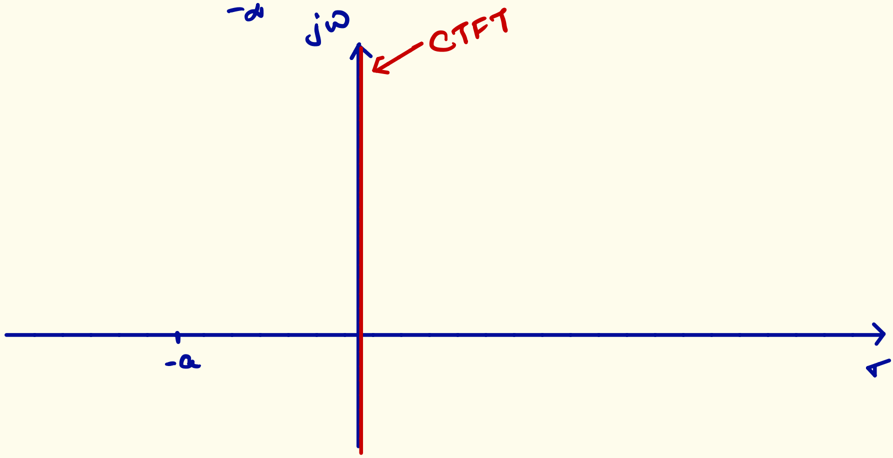
# Bilateral Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad \text{in some ROC}$$

$s = \sigma + j\omega$

## CTFT

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$



$$X(s) = X(\sigma + j\omega)$$

$\sigma = 0 \qquad X(j\omega) = X(\omega)$ if ROC includes $j\omega$ axis.

eg1. $x(t) = e^{-at} u(t)$

$$X(s) = \frac{1}{s+a} \qquad Re(s) > -a$$

$\rightarrow \int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$

$$X(\omega) = \frac{1}{j\omega + a} = X(j\omega)$$ as ROC includes $j\omega$ axis.

**eg 2.** $x(t) = u(t)$

$X(s) = \frac{1}{s}$ $\quad Re(s) > 0$ → does not include $j\omega$ axis

$$X(\omega) = \boxed{\pi\delta(\omega) +} \frac{1}{j\omega}$$

$$X(j\omega) \neq X(\omega)$$

System Analysis → Laplace Transform

Signal Analysis → Fourier Transform

| Time Domain $x(t)$ | Frequency Domain $X(\omega)$ |
|---|---|
| 1. Multiplication | 1. Convolution |
| 2. Convolution | 2. Multiplication |
| 3. Contraction | 3. Expansion |
| 4. Expansion | 4. Contraction |
| 5. Time Limited ⇒ | 5. Band Unlimited |
| 6. Discreteness | 6. Periodicity |
| 7. Periodicity | 7. Discreteness |
| 8. Delay / Advance | 8. Linear Phase |
| 9. Time Unlimited | ⇐ 9. Band Limited |
| 10. Impulse Response $h(t)$ | 10. Frequency Response $H(\omega)$ |

# Discrete Time Signals and Systems

DT Signals

- Intrinsically discrete

  eg. Monthly Income, Tax per year, stock closing value each day, Rainfall per day

- Sampling Continuous Time Signals.

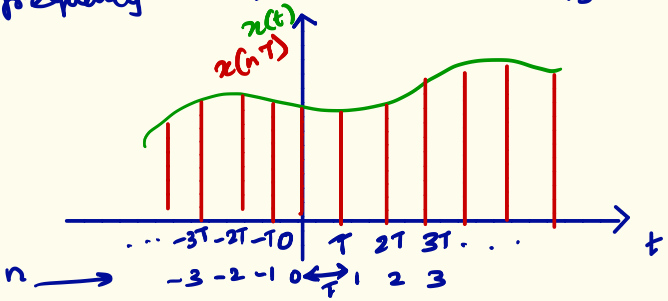
$x(t) \longrightarrow$ Essential BW $= W_e$ Hz

$t \in \mathbb{R}$

$\underset{n \in \mathbb{Z}}{x(nT)}$, $T \rightarrow$ sample spacing (sec)

Nyquist Theorem $T \leq \frac{1}{2W_e}$

Sampling frequency $f_s \geq 2W_e$ $\quad f_s = \frac{1}{T}$



$T \leq \frac{1}{2We}$

Continuous Time, Analog → Analog

Discrete Time, Digital → Digital

$x(nT)$ , $n \in \mathbb{Z}$

$x[n]$ , $n \in \mathbb{Z}$

- Energy
- Power

] × both

Energy Signals

$$E_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2 , \quad x[n] \to 0 \text{ as } |n| \to \infty$$

$\Rightarrow E_x$ is finite

Power Signals

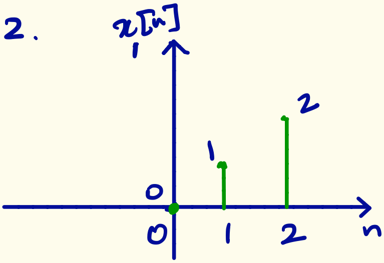
$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 \to \text{finite}$$

Period = $2N+1$

eg.

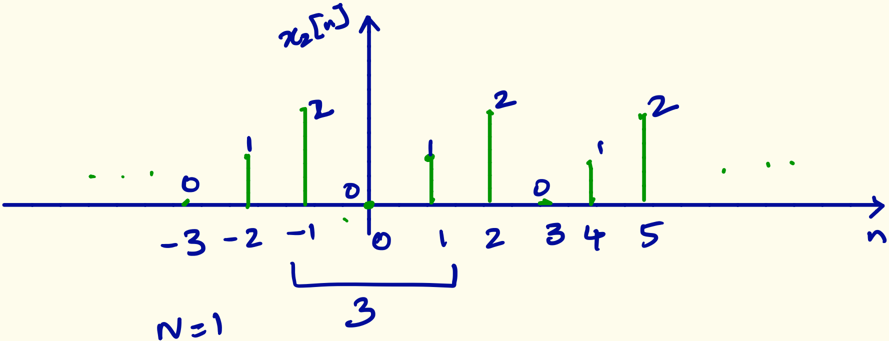
$x_1[n] = n \quad , \quad n = 0,1,2.$

$E_{x_1} = 0^2 + 1^2 + 2^2 = 5$



$x_2[n] = n, \quad n = 0,1,2$

$x_2[n+3k] = x_2[n] \quad \forall k \in \mathbb{Z}$



N=1

$$P_x = \frac{1}{3} \sum_{n=-1}^{+1} |x[n]|^2$$

$$= \frac{1}{3}\left[0^2 + 1^2 + 2^2\right]$$

$$= \frac{5}{3}$$

# Signal Operations

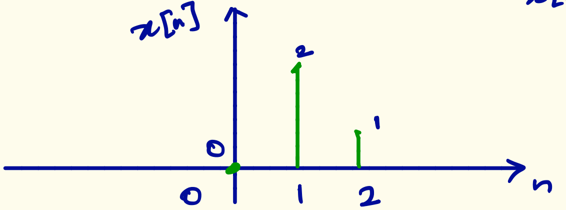
## 1. Delay / Advance

$x[n]$

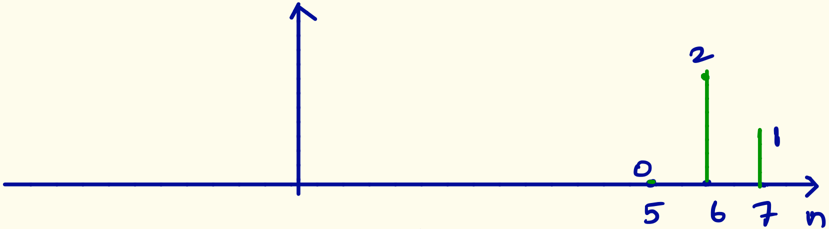
$x_d[n] = x[n - \underset{\substack{\uparrow \\ \text{integer}}}{\underline{5}}] \longrightarrow$ delayed version of $x[n]$ by 5 units.

$x_a[n] = x[n+2] \longrightarrow$ Advanced version of $x[n]$ by 2 units.

eg:



$x_d[n] = x[n-5]$



$x_a[n] = x[n+2]$

## 2. Time Inversion

$$x_r[n] = x[-n]$$

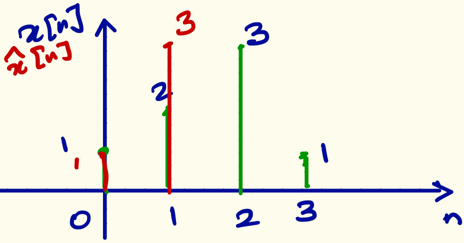
Reflection across amplitude axis.

## 3. Time Scaling

- Decimation | Downsampling

$$x[n]$$

$$\hat{x}[n] = x[2n]$$



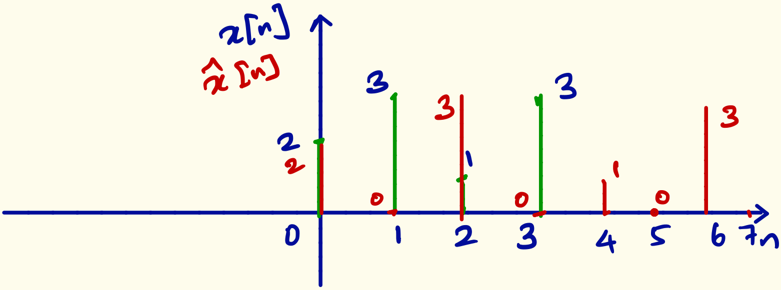
$\hat{x}[0] = x[0]$, $\hat{x}[1] = x[2]$, $\hat{x}[2] = 0$

$\hat{x}[n] = x[Mn]$ for any +ve integer M

→ $\hat{x}[n]$ retains only the every $M^{th}$ sample of $x[n]$

- Upsampling

$$\hat{x}[n] = x[n/2]$$



$\hat{x}[0] = x[0]$, $\hat{x}[1] = 0$, $\hat{x}[2] = x[1]$, $\hat{x}[3] = 0$, $\hat{x}[4] = x[2]$

$\hat{x}[5] = 0$, $\hat{x}[6] = x[3]$

$\hat{x}[n] = x[n/L]$ L → any +ve integer

$\hat{x}[n]$ retains values of $x[n]$ in every $L^{th}$ sample.

$\hat{x}[n] = 0$ in other samples.

## Application

Image Super-resolution

$\hat{x}[n] = x[an+b]$ , $a, b$ → integers

1. Shift $x[n]$ by $b$ samples

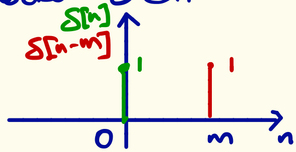
2. Scale and/or invert based on 'a'.

## Basic DT signals

1. Unit Sample

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{o/w} \end{cases}$$

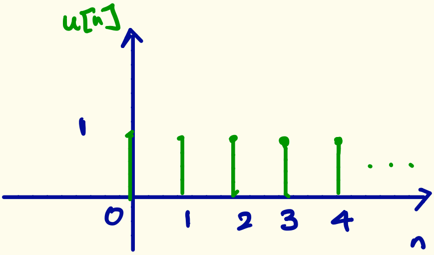
Kronecker Delta



$$\delta[n-m] = \begin{cases} 1, & n=m \\ 0, & \text{o/w} \end{cases}$$

2. Unit step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$\delta[n] = u[n] - u[n-1] \qquad \delta(t) = \frac{du(t)}{dt}$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k], \; n=0,1,2,\ldots \quad u[0] = \sum_{k=0}^{\infty} \delta[0-k] = \overset{k=0}{1}$$

## 3. Complex Exponential

$$e^{\lambda n} \quad , \quad n \in \mathbb{Z}$$

$$e^{\lambda} = \gamma \quad , \quad \begin{aligned} &re\,\lambda = 0 \Rightarrow e^{0} = 1 = \gamma \\ &re\,\lambda < 0 \Rightarrow |e^{-ve}| < 1 \\ &re\,\lambda > 0 \Rightarrow |e^{+ve}| > 1 \end{aligned}$$

$$\underline{e^{\lambda n}} = \underline{\gamma^{n}}$$

$$e^{\lambda} = e^{\sigma} \cdot e^{j\Omega}$$

**Case 1** $\sigma = 0$

$$|e^{\lambda}| = |e^{0}|\,|e^{j\Omega}| = 1 = |\gamma|$$

**Case 2** $\sigma < 0$

$$|e^{\lambda}| = \underset{\overset{\uparrow}{|\gamma|}}{|e^{\sigma}|} < 1$$

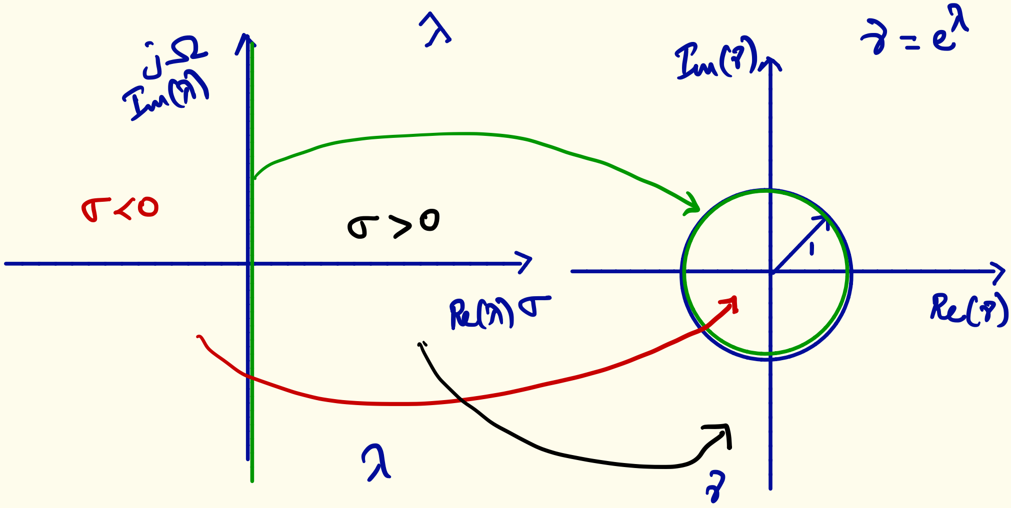
$$|e^{j\Omega}|$$

$$= |\cos\Omega + j\sin\Omega|$$

$$= 1$$

**Case 3** $\sigma > 0$

$$|e^{\lambda}| = \underset{\overset{\uparrow}{|\gamma|}}{|e^{\sigma}|} > 1$$



Sinusoids → $j\Omega$ axis or unit circle

Decreasing Exponentials → LHP or inside unit circle

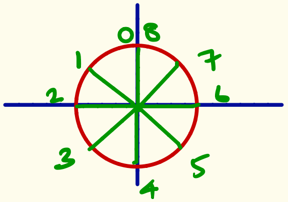
Increasing Exponentials → RHP or Outside unit circle

## 4. Discrete Sinusoids

$$x[n] = \underset{\text{Amplitude}}{C} \cos(\underset{\text{radians/sample}}{\Omega} \overset{\text{sample}}{n} + \underset{\text{Phase in radians}}{\theta})$$

$$2\pi F = \Omega \quad , \quad F = \frac{\Omega}{2\pi} \text{ Cycles/sample}$$

eg.



$$\underline{x_1[n] = \cos\left(\frac{\pi}{4}n + \frac{\pi}{2}\right)}$$

$\Omega = \frac{\pi}{4}$ rad/sample   Period $N_0 = \overset{m=1}{\frac{1}{F}} = 8$ samples/cycle

$F = \frac{1}{8}$ cycles/sample

$$x_2[n] = \cos\left(\frac{n}{4} + \frac{\pi}{2}\right)$$

$\Omega = \frac{1}{4}$ rad/sample   $N_0 = \frac{\boxed{m}}{\boxed{F}}$ → for integer

$F = \frac{1}{8\pi}$ cycles/sample   $= 8\boxed{\pi}\boxed{m}$

- All DT sinusoids are not periodic
- F → rational for periodicity.

$$\cos(\Omega n) = \cos(\Omega(n+N_0))$$ for some integer $N_0$

$$= \cos(\Omega n + \underline{\Omega N_0})$$

chapter 9

$$\underline{\Omega \underline{N_0}} = 2\pi m \quad , \quad m \rightarrow \text{integer}$$

$$\cos(\Omega n + 2\pi m) = \cos(\Omega n)$$

$$\Omega = 2\pi\left(\frac{m}{N_0}\right)$$

$$N_0 = 2\pi\left(\frac{m}{\Omega}\right)$$

$$N_0 = \frac{m}{F}$$ , for integer $m$

Conditions for periodicity

1. $\Omega \rightarrow 2\pi$ times rational

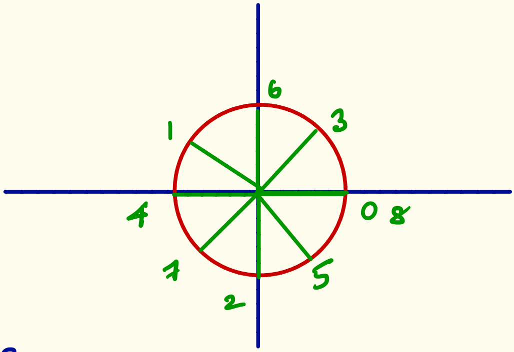
2. $F \rightarrow$ rational

3. $N_0 \rightarrow$ integer

No. of samples required to reach $2\pi m$ $(\Omega N_0 \rightarrow 2\pi m)$

eg.

$$\cos\left(\underbrace{\frac{3\pi}{4}}_{\Omega} n\right)$$



$$N_0 = 2\pi \frac{m}{\Omega}$$

$$= 4\times\frac{2\pi}{3\pi} \cdot m = \frac{8}{3} m$$ ↖ smallest integer

$$m=3$$

## 5. Complex Exponential

$$e^{j\Omega n} = \cos \Omega n + j \sin \Omega n$$

$|e^{j\Omega n}| = 1$ → lies on unit circle

Position → $\Omega n$

# Discrete Time Systems

1. Linear Vs Non-linear

2. Time Variant Vs Time Invariant

3. Causal Vs Non-causal

4. Memory Vs Memory less

5. Stable Vs Unstable

6. Invertible Vs Non-invertible

DT LTI Causal

eg. 1 Bank Balance

$y[n]$ → Balance at $n$ (including deposit at $n$)

$x[n]$ → Deposit made at $n$

$r$ → Interest Rate.

$$y[n] = \overbrace{y[n-1]}^{\text{Prev. Balance}} + \overbrace{r\,y[n-1]}^{\text{Interest}} + \overbrace{x[n]}^{\text{Deposit}}$$

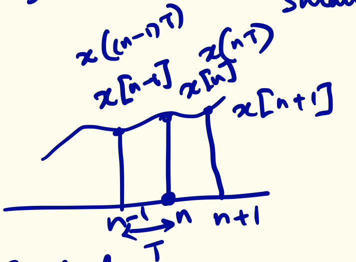
**eg 2** Differentiator

$$y(t) = \frac{dx(t)}{dt}$$

$$y(nT) = \lim_{T \to 0} \frac{x(nT) - x((n-1)T)}{T}$$

BW Difference $y[n] = \dfrac{x[n] - x[n-1]}{T}$ , $T \to 0$ (very small)

FW Difference $y[n] = \dfrac{x[n+1] - x[n]}{T}$



Central Difference

$$y[n] = \frac{x[n+1] - x[n-1]}{2T}$$

**eg. 3** Integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$y(nT) = \sum_{k=-\infty}^{n} x(kT) \cdot T$$

set

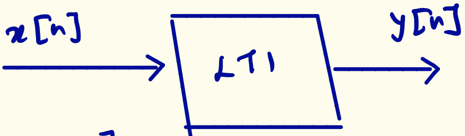
$$y[n] = T \sum_{k=-\infty}^{n} x[k] \text{ ——— } ①$$

$$y[n-1] = T \sum_{k=-\infty}^{n-1} x[k] \text{ ——— } ②$$

$$y[n] - y[n-1] = T\, x[n]$$

# DT LTI Systems

— Constant Coefficient Difference equations.



Current sample

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N]$$
$$= b_{N-M} x[n] + b_{N-M+1} x[n-1] + \dots + b_N x[n-M]$$

①

$$y[n+N] + a_1 y[n+N-1] + \dots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \dots + b_{N-1} x[n+1] + b_N x[n]$$

②

If $M > N$, $y[n+N]$ depends on $x[n+M]$ → Non-causal

for $M = N$ (Generalized Causal) Causal $\Rightarrow M \leq N$

① $\Rightarrow$

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \dots + b_N x[n-N]$$

③

② $\Rightarrow$

$$y[n+N] + a_1 y[n+N-1] + \dots + a_N y[n]$$
$$= b_0 x[n+N] + b_1 x[n+N-1] + \dots + b_N x[n]$$

④

eg. **Recursive Method**

$$y[n] = y[n-1] + 3x[n]$$

$y[-1] = 4$ , $x[n] = 0.5^n$

Total Response

$n=0$ $\quad y[0] = y[-1] + 3x[0]$

$$= 4 + 3 \times 1$$

$$= 7$$

$$y[1] = y[0] + 3x[1]$$

$$= 7 + \frac{3}{2}$$

$$= \frac{17}{2}$$

$\vdots$

Causal LTI system

$$\underbrace{y[n]}_{\text{Total}} = \underbrace{y_0[n]}_{\text{zero Input}} + \underbrace{y_{zs}[n]}_{\text{zero State}}$$

$E \rightarrow$ Operator

$$E\,y[n] = y[n+1] \qquad E\,x[n] = x[n+1]$$

$$E^2 y[n] = y[n+2] \qquad E^2 x[n] = x[n+2]$$

$$\vdots \qquad\qquad \vdots$$

$$E^N y[n] = y[n+N] \qquad E^N x[n] = x[n+N]$$

④ $\Rightarrow$

$$\left(E^N + a_1 E^{N-1} + \dots + a_N\right) y[n]$$

$$= \left(b_0 E^N + b_1 E^{N-1} + \dots + b_N\right) x[n]$$

$$Q(E)\, y[n] = P(E)\, x[n] \qquad D \rightarrow \frac{d}{dt}$$

**Zero Input Response** $y_0[n]$

Solution of

$$Q(E)\, y_0[n] = 0$$

$$y_0[n] = c\gamma^k$$

$$E\,\gamma^k = \gamma \cdot \gamma^k \qquad E^N \gamma^k = \gamma^N \cdot \gamma^k$$

$$E^2 \gamma^k = \gamma^2 \cdot \gamma^k$$

$$\vdots$$

$$Q(E)\, c\gamma^k = 0$$

$$(\gamma^N + a_1\gamma^{N-1} + \dots + a_N)\, \underline{c\gamma^k} = 0$$

For non-trivial solution

$$\gamma^N + a_1\gamma^{N-1} + \dots + a_N = 0$$

$$Q(\gamma) = 0$$

$$(\gamma - \gamma_1)(\gamma - \gamma_2)\dots(\gamma - \gamma_N) = 0$$

$\gamma_1, \gamma_2, \dots, \gamma_N$ → characteristic roots

$\gamma_1^n, \gamma_2^n, \dots, \gamma_N^n$ → characteristic modes

**Case 1** $\gamma_i$'s are distinct and real.

$$y_0[n] = c_1\gamma_1^n + c_2\gamma_2^n + \dots + c_N\gamma_N^n$$

**Case 2** $\gamma_r$ → repeated '$r$' times, real

$$y_0[n] = (c_1 + c_2 n + c_3 n^2 + \dots + c_r n^{r-1})\gamma_r^n + c_{r+1}\gamma_{r+1}^n + \dots + c_N\gamma_N^n$$

**Case 3** $\gamma_1, \gamma_2$ are complex conjugates

$\gamma_1 = |\gamma| e^{j\beta}$ $\gamma_2 = |\gamma| e^{-j\beta}$

$$y_0[n] = c|\gamma|^n \cos(\beta n + \theta) + c_3\gamma_3^n + \dots + c_N\gamma_N^n$$

eg. 1. $y[n+2] + 2y[n+1] + y[n] = 3x[n]$

$$(E^2 + 2E + 1)\, y[n] = 3x[n]$$

$$y_0[-2] = 1, \quad y_0[-1] = 3 \xrightarrow{0} \text{Initial Conditions}$$

$$y_0[n] = (c_1 + c_2 n)(-1)^n$$

$$y_0[n] = (-7 + -4n)(-1)^n$$

eg. 2

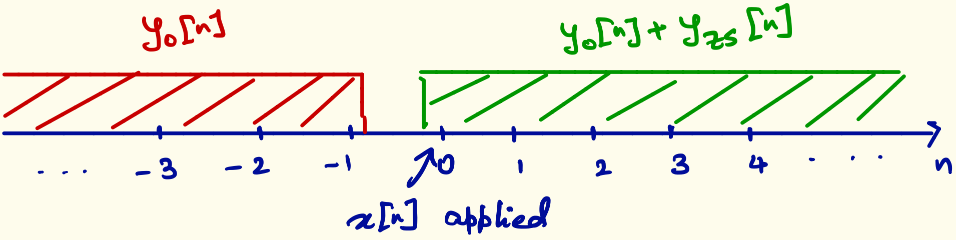
$$(E^2 + 5E + 6)\, y[n] = (E^2 + 3)\, x[n]$$

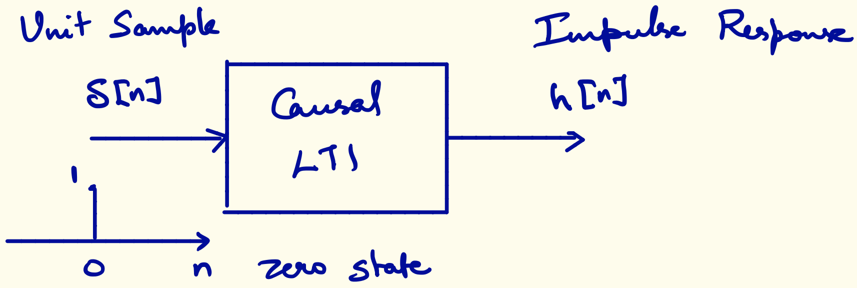
$$y_0[-2] = 1, \quad y_0[-1] = 2$$

$$y_0[n] = c_1 (-2)^n + c_2 (-3)^n$$

$$y_0[n] =$$

$$y[n] = y_0[n] + y_{zs}[n]$$





$$\ldots = h[-3] = h[-2] = h[-1] = 0 \qquad (\text{causal})$$

$$Q(E)\,y[n] = P(E)\,x[n]$$

$$x[n] = \delta[n], \qquad y[n] = h[n]$$

$$Q(E]\,h[n] = P(E)\,\delta[n]$$

$$h[n] = A_0\delta[n] + \underline{y_c[n]}\,u[n]$$

$$Q(E)\,y_c[n]\,u[n] = 0 \rightarrow \text{characteristic modes}$$

$$Q(E)\,A_0\,\delta[n] = P(E)\,\delta[n]$$

$$A_0\left[\delta[n+N] + a_1\delta[n+N-1] + \cdots + a_{N-1}\delta[n+1] + a_N\delta[n]\right]$$

$$= b_0\delta[n+N] + b_1\delta[n+N-1] + \cdots + b_N\delta[n]$$

When $n=0$

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & o/w \end{cases}$$

$$A_0 \, a_N \, \delta[0] = b_N \, \delta[0]$$

$$A_0 \, a_N = b_N$$

$A_0 = 0$ when $b_N = 0$

$$A_0 = \frac{b_N}{a_N}$$

$$h[n] = \frac{b_N}{a_N} \delta[n] + y_c[n] \, u[n]$$

<u>eg.</u>

$$(E^2 + 3E + 2) \, y[n] = 4 x[n]$$

LTI Causal

$$h[n] = y_c[n] \, u[n]$$

$h[-2] = 0$

$h[-1] = 0$

$$(E^2 + 3E + 2) \, h[n] = 4 \delta[n]$$

$$h[n+2] + 3h[n+1] + 2h[n] = 4 \delta[n]$$

$n = -2$

$h[0] = 0$

$n = -1$

$h[1] = 0$

$n = 0$

$h[2] = 4$

$n = 1$

$h[3] = -3h[2] - 2h[1]$

$h[3] = -12$

$n = 2$

$h[4] = -3h[3] - 2h[2]$

$h[4] = 28$

## Impulse Response

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

LTI Causal System

$$y[n+2] - 0.6\,y[n+1] - 0.16\,y[n] = 5x[n+2]$$

$$Q(E)\,y[n] = P(E)\,x[n]$$

$a_N = -0.16$

$b_N = 0$

$$Q(E) = E^2 - 0.6E - 0.16$$

$$P(E) = 5E^2$$

$Q(\gamma) = 0$ solutions are characteristic roots.

$$\gamma^2 - 0.6\gamma - 0.16 = 0$$

$$(\gamma - 0.8)(\gamma + 0.2) = 0$$

$$\gamma = 0.8,\ -0.2$$

$$y_c[n] = c_1(0.8)^n + c_2(-0.2)^n \Rightarrow h[n] = [c_1(0.8)^n + c_2(-0.2)^n]u[n] \quad ①$$

$\underline{h[-1]=0}$, $\underline{h[-2]=0}$, $\delta[0]=1$

$n=-2$, $x[n] = \delta[n]$, $y[n] = h[n]$

$$h[0] - 0.6\,h[-1] - 0.16\,h[-2] = 5\,\delta[0] \Rightarrow h[0] = 5$$

$n=-1$

$$h[1] - 0.6\,h[0] - 0.16\,h[-1] = 5\,\delta[1] \Rightarrow h[1] = 3$$

$h[0]$, $h[1]$ in ①

$$5 = c_1 + c_2$$

$$3 = 0.8c_1 - 0.2c_2$$

$c_1 = 4$, $c_2 = 1$

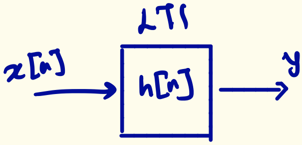
$$\Rightarrow \quad h[n] = \left[4(0.8)^n + (-0.2)^n\right] u[n]$$

Total Response

$$y[n] = y_0[n] + y_{zs}[n]$$

$$y[n] = y_0[n] + x[n] * h[n]$$

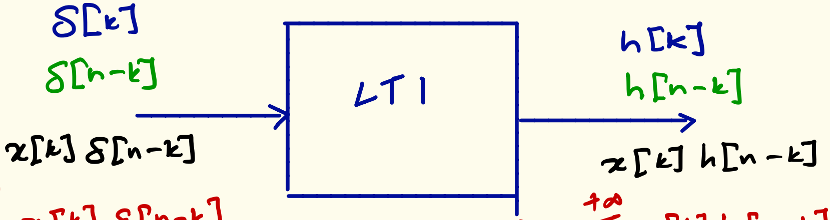
## Zero-State Response



$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\, \delta[n-k] \longrightarrow$$ Weighted Sum of shifted unit impulses.

$$\delta[n-k] = \begin{cases} 1, & n=k \\ 0, & n \neq k \end{cases}$$



$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\, \delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

## Properties

1. Commutative $\quad x_1[n] * x_2[n] = x_2[n] * x_1[n]$

2. Associative

$$(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n])$$

3. Distributive

$$x_1[n] * \left[x_2[n] + x_3[n]\right] = x_1[n] * x_2[n] + x_1[n] * x_3[n]$$

4. Delay

$$c[n] = x_1[n] * x_2[n]$$

$$x_1[n-k_1] * x_2[n-k_2] = c[n-k_1-k_2]$$

5. $$x[n] * \delta[n] = x[n]$$

6. Width

width

$$x_1[n] \longrightarrow L_1$$

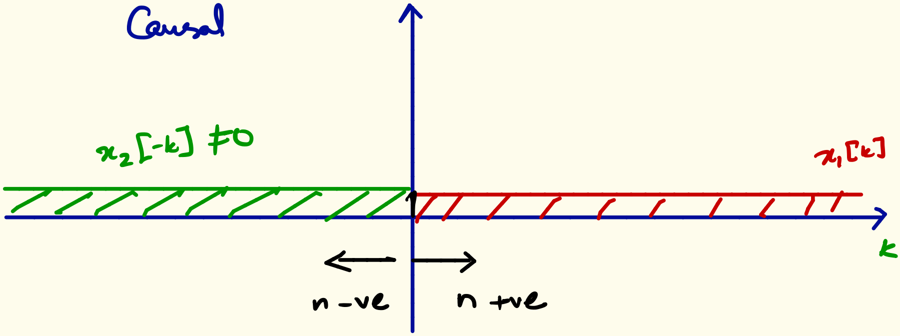
$$x_2[n] \longrightarrow L_2$$

$$x_1[n] * x_2[n] \longrightarrow L_1 + L_2 - 1$$

7. Causal → $x_1[n]$ & $x_2[n]$

$$x_1[n] * x_2[n] = \sum_{k=0}^{n} x_1[k]\, x_2[n-k]$$

Causal



eg. Tape Method

$x_1[n]$ $L_1 = 6$

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2<br>↑ | 3 | -1 | 4 | 1 | 3 |
| n | 0 | 1 | 2 | 3 | 4 | 5 |

$x_2[n]$ $L_2 = 3$

| | | | |
|---|---|---|---|
| | 3<br>↑ | 1 | 2 |
| n | 0 | 1 | 2 |

$$y[n] = \sum_{k=0}^{n} x_1[k]\, x_2[n-k]$$ as $x_1[n], x_2[n]$ Causal

$x_1[k]$

2 3 -1 4 1 3
↑
0

$x_2[-k]$

2 1 3
↑
0

$y[0] = 6$

$x_2[1-k]$

2 1 3
↑
0

$y[1] = 11$

$x_2[2-k]$

2 1 3
↑
0

$y[2] = 4$

$x_2[3-k]$

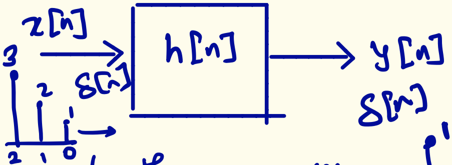
0 2 1 3
↑
0

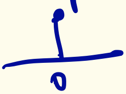
$y[3] =$

$L_1 + L_2 - 1$ inner products

8

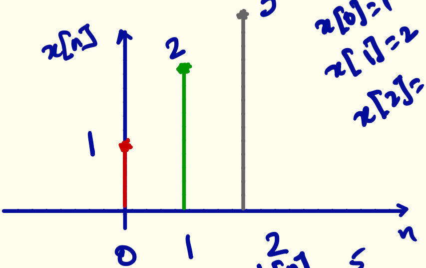
# A Discrete Example



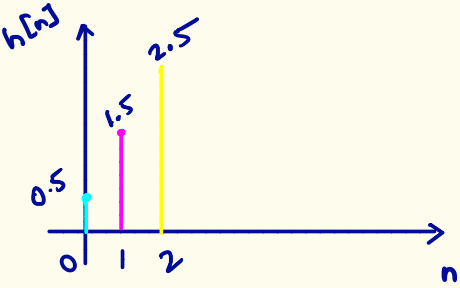


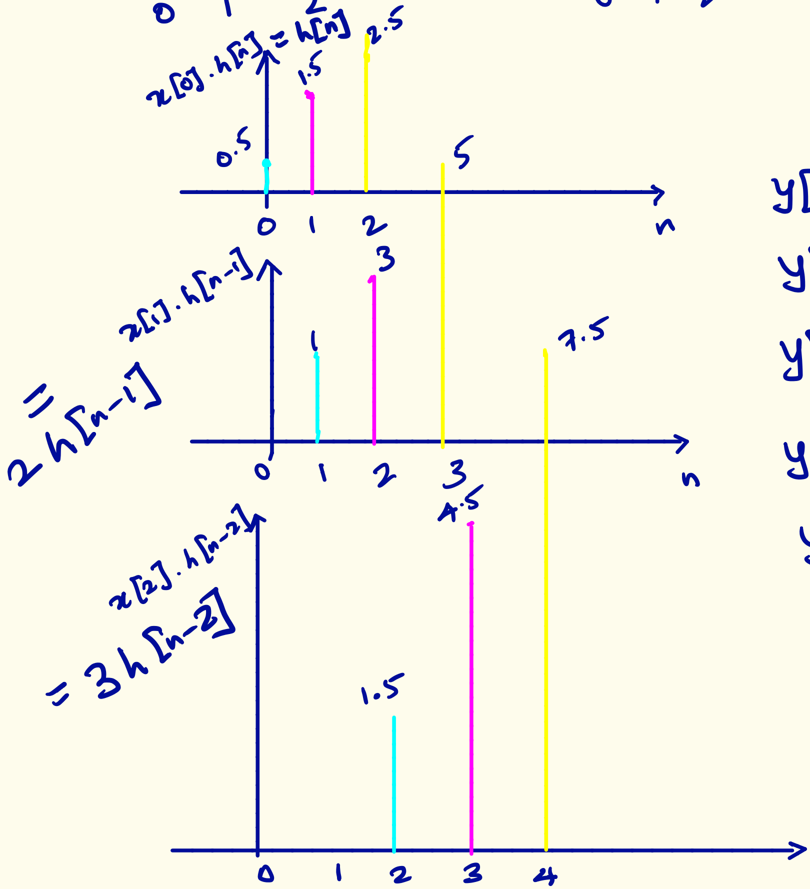
x[n] , h[n] → Finite length sequences.

①



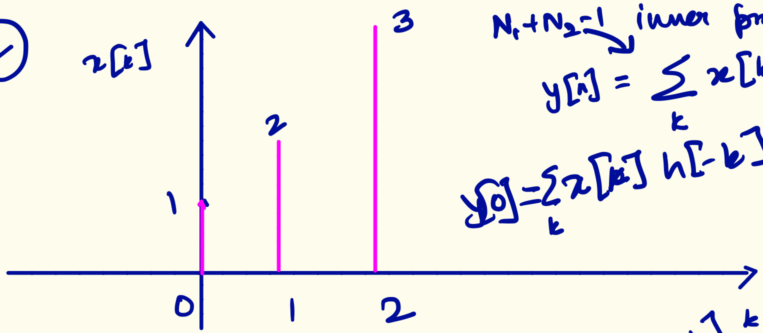
x[0]=1
x[1]=2
x[2]=3





y[0] = 0.5
y[1] = 2.5
y[2] = 7
y[3] = 9.5
y[4] = 7.5

②



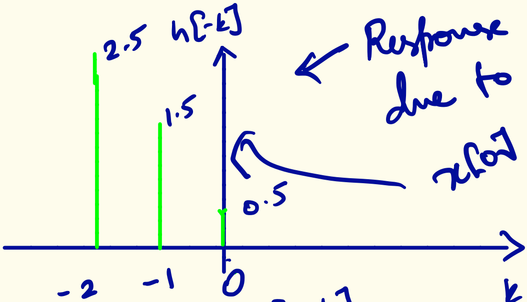
$N_1+N_2-1$ inner products

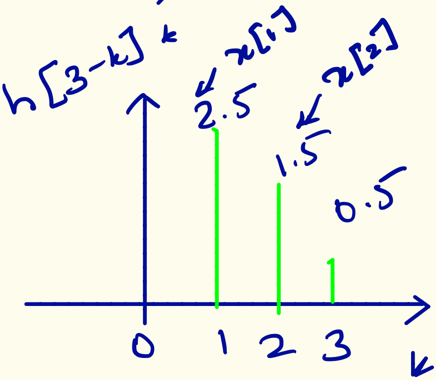
$$y[n] = \sum_k x[k]\, h[n-k]$$

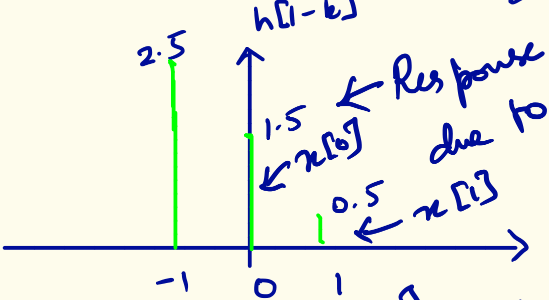
$$y[0] = \sum_k x[k]\, h[-k]$$

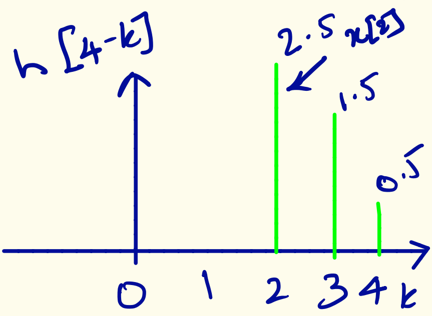
$x[n] \rightarrow N_1$

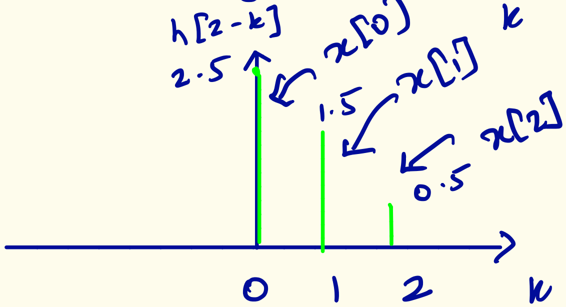
$h[n] \rightarrow N_2$











$y[0] = x[k] \cdot h[-k]$

$= 0.5$

$y[1] = x[k] \cdot h[1-k]$

$= 2.5$

$y[n] = N_1 + N_2 - 1$

$y[2] = 7$, $y[3] = 9.5$ $y[4] = 7.5$

**eg.** $x[n] = (0.8)^n u[n],\ g[n] = (0.3)^n u[n] \rightarrow$ Causal

$$c[n] = x[n] * g[n]$$

$$= \sum_{k=0}^{n} (0.8)^k \cdot (0.3)^{n-k}$$

Finite GP

$$= (0.3)^n \sum_{k=0}^{n} \left(\frac{0.8}{0.3}\right)^k \rightarrow$$

$a = 1$

$r = \left(\frac{0.8}{0.3}\right)$

$$\frac{a(r^{n+1}-1)}{r-1}$$

$$= (0.3)^n \frac{\left(\frac{0.8}{0.3}\right)^{n+1} - 1}{\left(\frac{0.8}{0.3}\right) - 1}$$

$$= \cancel{(0.3)^n . 0.3} \frac{(0.8)^{n+1} - (0.3)^{n+1}}{\cancel{(0.3)^{n+1}}(0.8 - 0.3)}$$

$$= 2\left[0.8^{n+1} - 0.3^{n+1}\right] u[n]$$

$$y[n+2] - 0.6\,y[n+1] - 0.16\,y[n] = 5\,x[n+2]$$

**eg.** $y[n] - 0.6\,y[n-1] - 0.16\,y[n-2] = 5\,x[n]$

$y_0[-2] = 1,\ y_0[-1] = 3$

$x[n] = (0.25)^n u[n]$

$$y[n] = \overset{\text{zero Input}}{y_0[n]} + \overset{\text{zero state}}{x[n] * h[n]}$$

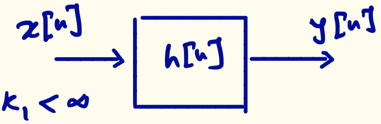
$$= 7.96(0.8)^n + 1.39(-0.2)^n + (0.25)^n u[n] * \left(\left[4(0.8)^n + (-0.2)^n\right] u[n]\right)$$

# Stability of LTID Systems (Causal)

1. Zero-state stability - BIBO

2. Zero-input stability - Asymptotic

## 1. BIBO



Bounded Input $|x[n]| < k_1 < \infty$

$$|y[n]| = \left|\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]\right| < \sum_{k=-\infty}^{+\infty} |x[k]|\,|h[n-k]|$$

$$< k_1 \sum_{k=-\infty}^{+\infty} |h[n-k]|$$

$$|y[n]| < \infty$$

$|y[n]| < k_1 k_2$ where $k_1 < \infty$, $k_2 < \infty$

Therefore $\displaystyle\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

For causal $\displaystyle\sum_{k=0}^{\infty} |h[k]| < \infty$

Impulse Response → Absolutely summable

$\Rightarrow$ BIBO stable

## 2. Asymptotic Stability (Causal)

$y_0[n] \rightarrow$ zero input response

$$y_0[n] = \sum_{i=1}^{p} c_i \gamma_i^n$$

where $\gamma_i$'s are roots of $Q(\gamma) = 0$

$$\gamma = |\gamma| e^{j\beta}$$

$$\gamma^n = |\gamma|^n \underline{e^{j\beta n}}$$

<u>Case 1</u> $|\gamma_i| = 1 \Rightarrow |\gamma_i|^n = 1$ for some $i$

for others $|\gamma_i| < 1$

$\Rightarrow$ Marginally stable

<u>Case 2</u> $|\gamma_i| < 1$ for all $i$

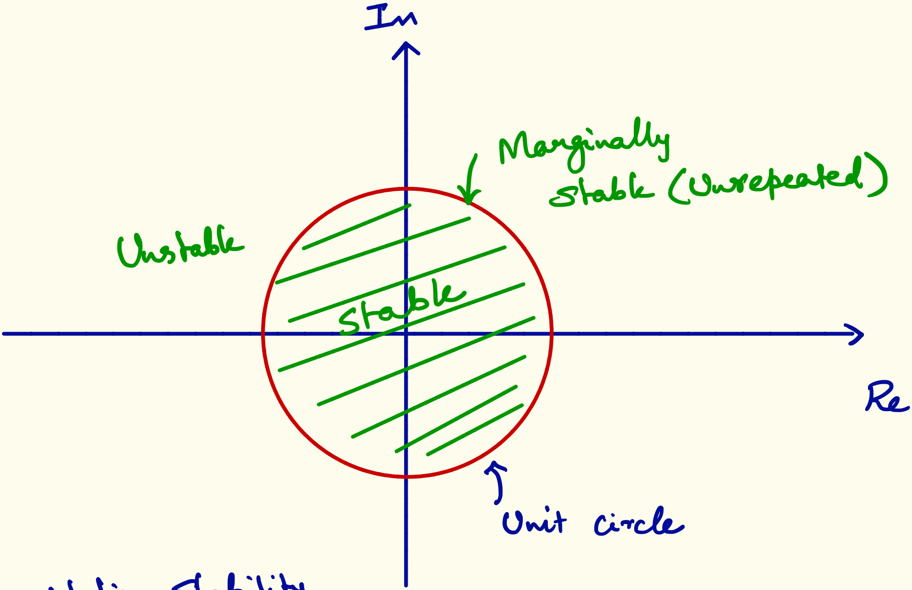
$c_i \gamma_i^n \rightarrow$ Exponentially Decreasing

$\Rightarrow$ Asymptotically stable $\Rightarrow$ BIBO stable

<u>Case 3</u> $|\gamma_i| > 1$ for some $i$

$c_i \gamma_i^n \rightarrow$ Exponentially Increasing

$\Rightarrow$ Asymptotically Unstable



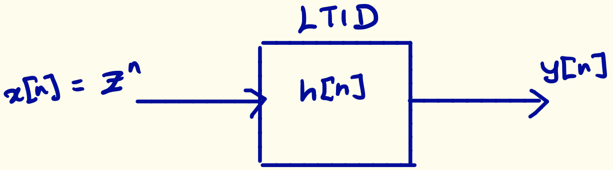
## Asymptotic Stability

An LTID Causal System is

1. Marginally stable iff all charac. roots ($\gamma_i$) lie inside the unit circle except few unrepeated on the unit circle.

2. Stable iff all char. roots ($\gamma_i$) lie inside the unit circle.

3. Unstable iff atleast one char. root lies outside the unit circle or atleast one repeated char. roots on the unit circle.



$z \in \mathbb{C}$

Zero-state Response

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\, z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k]\, z^{-k}$$

$$y[n] = z^n \; H[z]$$

$$H[z] = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} \longrightarrow \text{Transfer function}$$

If Causal

$$H[z] = \sum_{n=0}^{+\infty} h[n]\, z^{-n}$$

$$Q(E)\, y[n] = P[E]\, x[n]$$

$$Q(E)\, z^n H[z] = P[E]\, z^n$$

$$(E^N + a_1 E^{N-1} + \cdots + a_N) z^n H[z]$$

$$= (b_0 E^N + b_1 E^{N-1} + \cdots + b_N) z^n$$

$$(z^N + a_1 z^{N-1} + \cdots + a_N) z^n H[z]$$

$$= (b_0 z^N + b_1 z^{N-1} + \cdots + b_N) z^n$$

$$Q[z] z^n H[z] = P[z] z^n$$

$$H[z] = \frac{P[z]}{Q[z]} \Bigg| \text{ when } x[n] = z^n$$

## Z - Transform

Bilateral
$$X[z] = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X[z] z^{n-1} dz$$

## Unilateral Z-Transform

$$X[z] = \sum_{n=0}^{\infty} x[n]\, z^{-n} \qquad \text{(Causal)}$$

### Existence

$$|X[z]| < \infty$$

$$\sum_{n=0}^{\infty} x^n =$$

$$1 + x + x^2 + \dots$$

$$\Rightarrow \left| \sum_{n=0} x[n]\, z^{-n} \right| < \infty$$

$$= \frac{1}{1-x}, \quad |x| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \left| \frac{x[n]}{z^n} \right| < \infty$$

$$x[n] < \underline{r_0}^n$$

$$\left| \frac{r_0}{z} \right| < 1$$

$$\sum_{n=0}^{\infty} \left| \frac{r_0}{z} \right|^n$$

$$|z| > |r_0|$$

for any $x[n]$, if we can find $r_0$ such that

$$x[n] < r_0^n \Rightarrow Z \text{ Transform exists}$$

Any -Infinity signal $x[n]$ growing not faster than exponential has an Z-Transform $X[z]$.

Any finite $x[n]$ → Always has Z-Transform.

Eg.

1. $\delta[n-k] \xrightarrow{Z} z^{-k}$

$$\sum_{n=0}^{\infty} \delta[n-k]\, z^{-n} = z^{-k}$$

2. $u[n] \xrightarrow{Z} \frac{z}{z-1}$

$$\sum_{n=0}^{\infty} u[n]\, z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1}{1-\frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1 \qquad |z| > 1$$

$$= \frac{z}{z-1}$$

3. $\gamma^{n} u[n] \xrightarrow{Z} \frac{z}{z-\gamma}$

$$\sum_{n=0}^{\infty} \gamma^{n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^{n}$$

$$= \frac{1}{1-\frac{\gamma}{z}} \qquad |z| > |\gamma|$$

$$= \frac{z}{z-\gamma}$$

## Properties

1. Addition

$$x_1[n] + x_2[n] \xrightarrow{Z}$$

2. Scaling

$$a\,x[n] \xrightarrow{Z}$$

3. Right Shifting

$$x[n-m]\,u[n-m] \xrightarrow{Z}$$

$$x[n-m]\,u[n] \xrightarrow{Z}$$

$$\sum_{n=m}^{\infty} x[n-m]\, z^{-n}$$

$$n-m=r$$

$$= \sum_{r=0}^{\infty} x[r]\, z^{-(r+m)}$$

$$= z^{-m} \sum_{r=0}^{\infty} x[r]\, z^{-r} = z^{-m} X[z]$$

eg.

$$x[n] \quad 3 \quad 4 \quad \underset{\substack{\uparrow \\ 0}}{5} \quad 6 \quad 7$$

$$x[n-2]\,u[n-2]$$

$$\underset{\substack{\uparrow \\ 0}}{0} \quad 0 \quad \underset{\uparrow}{5} \quad 6 \quad 7$$

$$x[n-2]\,u[n] \checkmark$$

$$\underset{\substack{\uparrow \\ 0}}{3} \quad 4 \quad 5 \quad 6 \quad 7$$

$$Z\Big[x[n-m]\,u[n]\Big] = \sum_{n=0}^{\infty} x[n-m]\, z^{-n}$$

$$n-m=r$$

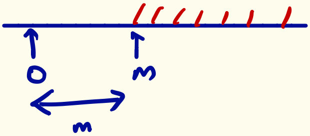
$$n=m+r$$

$$= \sum_{r=-m}^{\infty} x[r]\, z^{-(m+r)} = z^{-m}\left[\sum_{r=-m}^{-1} x[r]\, z^{-r} + \sum_{r=0}^{\infty} x[r]\, z^{-r}\right]$$

$$= z^{-m} X[z] + z^{-m} \sum_{r=-m}^{-1} x[r]\, z^{-r}$$

4. Left Shifting $x[n+m]\,u[n+m] \xrightarrow{Z} \underline{z^m X[z]}$

$x[n+m]\,u[n] \xrightarrow{Z}$



$$\sum_{n=0}^{\infty} x[n+m]\, z^{-n}$$

$n+m=r$

$$= \sum_{r=m}^{\infty} x[r]\, z^{-(r-m)} \overset{z^m}{=} \left[\sum_{r=0}^{\infty} x[r]\, z^{-r} - \sum_{r=0}^{m-1} x[r]\, z^{-r}\right]$$

$$= z^m X[z] - z^m \sum_{r=0}^{m-1} x[r]\, z^{-r}$$

$x[n+2]\,u[n+2]$ $\quad z^2 X[z]$

$$\begin{array}{ccccc} 0 & 0 & \underline{5} & \underline{6} & 7 \\ & & & & \uparrow \\ & & & & 0 \end{array}$$

$$\begin{array}{cccccc} x[n] & 3 & 4 & 5 & 6 & 7 \\ & & & \uparrow & & \\ & -2 & -1 & 0 & 1 & 2 \end{array}$$

$x[n+2]\,u[n]$

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 7 \\ & & & & \uparrow \\ & & & & 0 \end{array}$$

5. $\gamma^n x[n]\,u[n] \xrightarrow{Z} X\left[\frac{z}{\gamma}\right]$

$$\sum_{n=0}^{\infty} \gamma^n x[n]\, z^{-n} = \sum_{n=0}^{\infty} x[n] \left(\frac{z}{\gamma}\right)^{-n}$$

$$= X\left[\frac{z}{\gamma}\right]$$

6. $n\, x[n]\, u[n] \xrightarrow{Z} -z \frac{d}{dz}\left(X[z]\right)$

$$-z\frac{d}{dz}\left(X[z]\right) = -z \frac{d}{dz}\left(\sum_{n=0}^{\infty} x[n]\, z^{-n}\right)$$

$$= -\underline{z} \sum_{n=0}^{\infty} -n\, x[n]\, z^{-n-1}$$

$$= \sum_{n=0}^{\infty} \underline{n\, x[n]}\, z^{-n}$$

Time Convolution

7. $x_1[n] * x_2[n] \xrightarrow{Z} X_1[z]\, X_2[z]$

$$Z\left[x_1[n] * x_2[n]\right] = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{+\infty} x_1[k]\, x_2[n-k]\right) z^{-n}$$

$n-k=m$

$n = m+k$

$$= \sum_{m=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} \underline{x_1[k]}\, x_2[m]\, \underline{z}^{-(m+\underline{k})}$$

$$= \sum_{k=-\infty}^{+\infty} x_1[k]\, z^{-k} \sum_{m=-\infty}^{+\infty} x_2[m]\, z^{-k}$$

$$= X_1[z]\, X_2[z]$$

Time Reversal

8. $x[-n] \xrightarrow{Z} X\left[\frac{1}{z}\right]$

$$\sum_{n=-\infty}^{+\infty} x[-n]\, z^{-n}$$

$n = -m$

$$= \sum_{m=-\infty}^{+\infty} x[m]\, z^{m} = \sum_{m=-\infty}^{+\infty} x[m]\left(\frac{1}{z}\right)^{-m}$$

$$= X\left[\frac{1}{z}\right]$$

Initial Value

9. $x[0] = \lim_{z\to\infty} X[z]$

$X[z] = x[0] + \dfrac{x[1]}{z} + \ldots$

Final Value

10. $\lim_{N\to\infty} x[N] = \lim_{z\to 1} (z-1)\, X[z]$

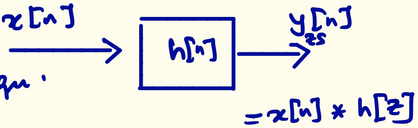
HW

## Stability

Transfer Function

$$H[z] = \frac{P[z]}{Q[z]}$$

Diff. equ.



$$H[z] = \sum_{n=0}^{\infty} h[n]\, z^{-n}$$ Impulse Response

$$Y_{zs}[z] = X[z]\, H[z]$$

$$H[z] = \frac{Y_{zs}[z]}{X[z]}$$ Zero state Response

Roots of $P[z]$ → zeros } of $H[z]$

Roots of $Q[z]$ → Poles } of $H[z]$

No common roots $P[z]$, $Q[z]$

LTID Causal System is

1. Asymptotically stable iff all the poles of $H[z]$ lie inside the unit circle.

2. Marginally stable iff all the poles of $H[z]$ lie inside the unit circle except unrepeated poles on the unit circle.

3. Unstable iff atleast one pole of $H[z]$ lies outside the unit circle or there are repeated poles on the unit circle.

eg.

1. $n\,u[n] \xrightarrow{Z} \dfrac{z}{(z-1)^2}$

$$-z\frac{d}{dz}\left(Z\left(u[n]\right)\right) = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\frac{\left[(z-1)-z\right]}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

2. $\gamma^n u[n] \xrightarrow{Z} \dfrac{z/\gamma}{\frac{z}{\gamma}-1} = \dfrac{z}{z-\gamma}$

$$X\left[\frac{z}{\gamma}\right] = $$

3. $n\gamma^n u[n] \xrightarrow{Z} -z\dfrac{d}{dz}\left(\dfrac{z}{z-\gamma}\right)$

$$= \frac{\gamma z}{(z-\gamma)^2}$$

4. $\cos\beta n\; u[n]$

$$= \left[\frac{e^{j\beta n}+e^{-j\beta n}}{2}\right]u[n]$$

$$= \frac{1}{2}\left[\left(e^{+j\beta}\right)^n u[n] + \left(e^{-j\beta}\right)^n u[n]\right]$$

$$\Big\downarrow Z$$

$$\frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right]$$

HW

$n\cos\beta n\; u[n]$

$n\gamma^n\cos\beta n\; u[n]$

LTID system (Causal) $H[z] = \frac{3z+5}{z^2-5z+6} \xrightarrow{z^{-1}} h[n]$

$\frac{P[z]}{Q[z]} \leftarrow$

1. $y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$

$\rightarrow y[-1] = 11/6 \quad , \quad y[-2] = \frac{37}{36} \quad , \quad x[n] = (0.5)^n u[n]$

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$$

$$Y[z] - 5\left[\frac{1}{z}Y[z] + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}\right] = z^{-1}\frac{3z}{z-0.5} + z^{-2}\frac{5z}{(z-0.5)} \quad ①$$

as $X[z] = \frac{z}{z-0.5}$

$$y[n-2]u[n] \xrightarrow{Z} \frac{1}{z^2}Y[z] + \frac{1}{z}y[-1] + y[-2] \qquad m=2$$

$$y[n-1]u[n] \xrightarrow{Z} \frac{1}{z}Y[z] + y[-1] \qquad m=1$$

$$x[n-m]u[n] \xrightarrow{Z} z^{-m}X[z] + z^{-m}\sum_{i=-m}^{-1} x[i]z^{-i}$$

$$x[n] = (0.5)^n u[n] \xrightarrow{Z} \frac{z}{z-0.5}$$

$$x[n-2] = (0.5)^{n-2}u[n-2] \xrightarrow{Z} z^{-2}\frac{z}{z-0.5}$$

$$x[n-1] = (0.5)^{n-1}u[n-1] \xrightarrow{Z} z^{-1}\frac{z}{z-0.5}$$

$$① \Rightarrow \left(Y[z] - \frac{5}{z}Y[z] + \frac{6}{z^2}Y[z]\right) + \left(\frac{11}{z} + \frac{37}{6} - \frac{55}{6}\right)$$

zero input

$$= \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

$$\left[\frac{z^2-5z+6}{z^2}\right]Y[z] = \left(-\frac{11}{z}+3\right)+\frac{3z+5}{z(z-0.5)}$$

$$Y_{zs}[z] = H[z]\,X[z] = \frac{3z+5}{z^2-5z+6}\cdot\frac{z}{z-0.5}$$

$$Y[z] = \frac{z^2\left(-\frac{11}{z}+3\right)}{z^2-5z+6} + \frac{\cancel{z^2}(3z+5)}{\cancel{z}(z^2-5z+6)(z-0.5)}$$

$$\frac{Y[z]}{z} = \frac{3z-11}{(z-3)(z-2)} + \frac{3z+5}{(z-3)(z-2)(z-0.5)}$$

$$\frac{Y[z]}{z} = \frac{5}{z-2} - \frac{2}{z-3} + \frac{5.6}{z-3} - \frac{22/3}{z-2} + \frac{1.733}{z-0.5}$$

$$Y[z] = \underbrace{\frac{5z}{z-2} - \frac{2z}{z-3}}_{Y_0[z]} + \underbrace{\frac{5.6z}{z-3} - \frac{22/3\,z}{z-2} + \frac{1.733z}{z-0.5}}_{Y_{zs}[z]}$$

$$y[n] = \underbrace{5(2)^n u[n] - 2(3)^n u[n]}_{y_0[n]} + \underbrace{5.6(3)^n u[n] - 7.33(2)^n u[n] + 1.733(0.5)^n u[n]}_{y_{zs}[n]}$$

$$y[n] = \underbrace{-2.33(2)^n u[n] + 3.6(3)^n u[n]}_{y_n[n]\ \text{Natural Response}} + \underbrace{1.733(0.5)^n u[n]}_{y_\phi[n]\ \text{Forced Response}}$$

2. $y[n] + 3y[n-1] + 2y[n-2] = x[n-1] + 3x[n-2]$

$x[n] = u[n]$ , $y[0] = 1$ , $y[1] = 2$

$y[n+2] + 3y[n+1] + 2y[n] = x[n+1] + 3x[n]$

$X[z] = \frac{z}{z-1}$ , $H[z] = \frac{z+3}{z^2+3z+2} = \frac{z+3}{(z+2)(z+1)}$

$= \frac{2}{z+1} - \frac{1}{z+2}$

$h[n] =$

<u>HW</u>

$$X[z] = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} \qquad z = \underline{r}\, e^{j\Omega}$$

Stable

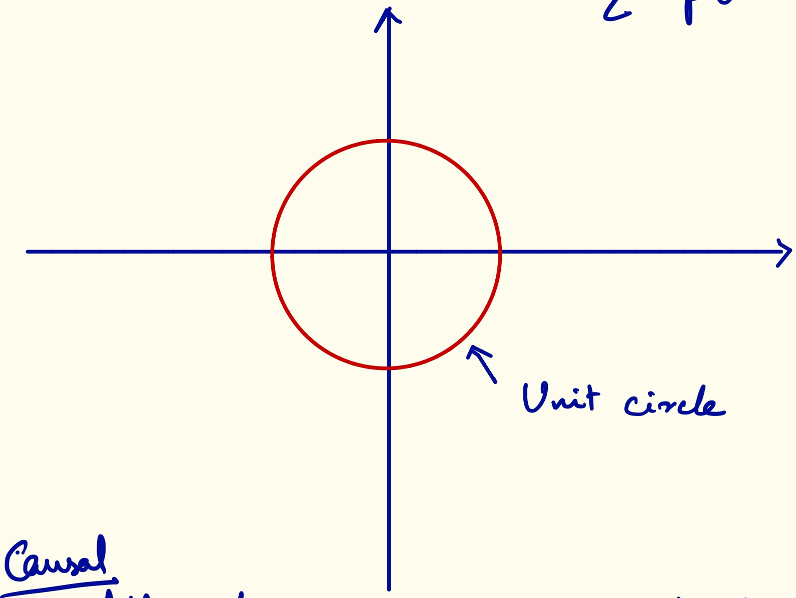
$r = 1$

$$X[e^{j\Omega}] = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n} \qquad \text{DTFT}$$

$$x[n] = \frac{1}{2\pi} \int_{\Omega=-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega \qquad \text{IDTFT}$$

$\underline{\Omega} \rightarrow$ Periodic in $2\pi$

Z-plane



**Causal**

All poles of $H[z]$ inside Unit circle $\Rightarrow$ ROC includes unit circle $\Rightarrow$ DTFT exists

All Systems (stable/Unstable) → Z transform

Stable Systems → DTFT

Discrete Time Signals → DTFT

Periodic DT Signals → DTFS

| Time | Freq. | |
|---|---|---|
| Discrete | Periodic | DTFS/<br>DFT<br>(fast DFT → FFT) |
| Periodic | Discrete | |